Matthew D. Bartos

Hydraulics Course Notes

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Contents

Dedicated to the UT Austin Water Resources Engineering Group

Introduction

This document contains the course notes for *CE356: Elements of Hydraulic Engineering* at the University of Texas at Austin, taught by Prof. Matthew Bartos.

Course overview

Elements of Hydraulic Engineering represents the fundamental course in hydraulic analysis and design that you will take during your civil engineering undergraduate curriculum. In fluid mechanics, you explored how force and energy balances could be used to analyze the behavior of simple fluid systems like pipes and tanks. In hydraulics, you will explore how these same fundamental principles can be applied to design complex fluid systems like pipeline and reservoir systems, water distribution networks, pump systems, and stormwater conveyance channels.

The course will focus on three main elements that comprise engineered water infrastructure: *pipes*, *pumps*, and *open channels*.

Figure 1: Pipes, pumps, and open channels.

Pipes: The first part of the course will focus on pipes and pipe systems. We will begin by reviewing the basic principles of conservation of mass, momentum, and energy that govern flow in pressurized pipe systems. We will cover common models for computing frictional losses in pipes, including Darcy-Weisbach, Hazen-Williams, and Chezy-Manning formulations. We will discuss sources of minor head losses and how to estimate them. Finally, we will discuss how to analyze and simulate flow in networked pipe systems such as those found in municipal water distribution networks.

Pumps: The second part of the course will focus on pumps, which are used to drive flow in pressurized pipe systems. We will discuss the major types of pumps, pump curves, and selection of pumps based on system head requirements. We will also discuss the behavior of variable-speed pumps through the use of affinity laws, and analyze the behavior of systems of pumps acting in parallel and in series.

Open channels: The third part of the course will focus on flow in open channels, which are integral within both urban drainage systems and large-scale water resources engineering projects like dams and conveyance infrastructure. We will discuss analysis of uniform flow conditions, analysis of transitions in flow using the energy equation, subcritical and supercritical flow, hydraulic jumps, gradually-varied flow profiles, and control structures like weirs and orifices.

The material covered in this class is central to understanding the engineered water cycle that we rely on to provide safe and reliable water for human use, to collect and treat water after use, and to manage water excess during storms. Pipe and pump systems are used extensively in water distribution networks, water and wastewater treatment facilities, and premise plumbing used by water consumers. Likewise, open channel flow governs the movement of water in wastewater and stormwater collection systems, natural waterbodies like rivers and lakes, and water supply and conveyance infrastructure.

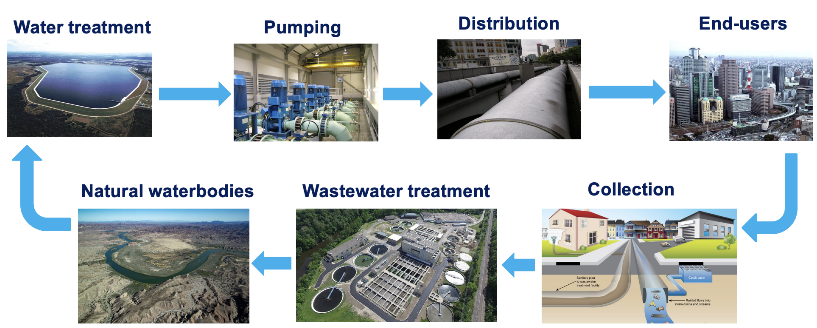


Figure 2: Engineered water cycle.

In this class, you will explore how fundamental physical knowledge obtained from fluid mechanics will allow you to design effective water management systems that meet the needs of each of these use cases.

Part I

Preliminaries

Properties of water

Review of properties of liquids

From fluid mechanics, you will recall that liquids like water have intrinsic properties that affect their physical behavior. Table 1 lists the major properties of fluids used in this class, along with their units (where M indicates mass, L indicates length, and T indicates time).

Property	Symbol	Description	Units
Mass density	ρ	Mass of liquid per unit volume	$[\frac{M}{L^3}]$
Specific weight	$\gamma = \rho g$	Weight of liquid per unit volume	$[\frac{M}{L^2 T^2}]$
Dynamic viscosity	μ	Measure of resistance to deformation by shear stress	$[\frac{M}{LT}]$
Kinematic viscosity	$\nu = \frac{\mu}{\rho}$	Dynamic viscosity divided by mass density	$[\frac{L^2}{T}]$
Modulus of elasticity	$E_v = -\frac{dp}{d\forall}\forall$	Change in volume due to change in pressure (i.e. compressibility)	$[\frac{M}{LT^2}]$
Vapor pressure	p_v	Pressure exerted by vapor on its liquid or solid form	$[\frac{M}{LT^2}]$

Table 1: Properties of liquids.

These physical properties are significantly affected by the temperature of the water. As temperature increases, the density, specific weight, and viscosity of water decrease, while vapor pressure increases with increasing temperature.

Tables of fluid properties by temperature

Density and specific weight: Table 2 shows the density and specific weight of water with respect to temperature in both SI and US units.

Vapor pressure: Table 3 shows the vapor pressure of water with respect to temperature in both SI and US units.

Viscosity: Table 4 shows the dynamic and kinematic viscosity of water with respect to temperature in both SI and US units.

Temperature		Density (ρ)		Specific weight (γ)	
°C	°F	kg/m^3	slug/ft^3	N/m^3	lb/ft^3
0 (ice)	32	917	1.78	8,996	57.3
0 (water)	32	999	1.94	9,800	62.4
4	39.2	1,000	1.94	9,810	62.4
10	50	999	1.94	9,800	62.4
20	68	998	1.94	9,790	62.3
30	86	996	1.93	9,771	62.2
40	104	992	1.92	9,732	62.0
50	122	988	1.92	9,692	61.7
60	140	983	1.91	9,643	61.4
70	158	978	1.90	9,594	61.1
80	176	972	1.89	9,535	60.7
90	194	965	1.87	9,467	60.3
100	212	958	1.86	9,398	59.8

Table 2: Density and specific weight of water

Temperature		Vapor pressure (v_p)	
°C	°F	$\times 10^3$ N/m^2	lb/in^2
-5	23	0.4210	0.06118
0	32	0.6110	0.08860
5	41	0.8730	0.1264
10	50	1.227	0.1779
15	59	1.707	0.2470
20	68	2.335	0.3387
25	77	3.169	0.4590
30	86	4.238	0.6149
35	95	5.621	0.8151
40	104	7.377	1.069
45	113	9.584	1.390
50	122	12.33	1.789

Table 3: Vapor pressure of water

Temperature		Dynamic viscosity (μ)		Kinematic viscosity (ν)	
°C	°F	$\times 10^{-3}\ N \cdot s/m^2$	$\times 10^{-5}\ lb \cdot s/ft^2$	$\times 10^{-6}\ m^2/s$	$\times 10^{-5}\ ft^2/s$
0	32	1.781	3.721	1.785	1.921
5	41	1.518	3.171	1.519	1.634
10	50	1.307	2.730	1.306	1.405
15	59	1.139	2.379	1.139	1.226
20	68	1.002	2.093	1.003	1.079
25	77	0.890	1.859	0.893	0.961
30	86	0.798	1.667	0.800	0.861
40	104	0.653	1.364	0.658	0.708
50	122	0.547	1.143	0.553	0.595
60	140	0.466	0.973	0.474	0.510
70	158	0.404	0.844	0.413	0.444
80	176	0.354	0.740	0.364	0.392
90	194	0.315	0.658	0.326	0.351
100	212	0.282	0.589	0.294	0.316

Table 4: Viscosity of water

Part II

Pressurized pipe flow

Flow in pipes

Overview of pressurized pipe flow module

The first portion of this class will focus on *pressurized pipe flow*. We will begin by reviewing basic properties of water and the fundamental laws that govern fluid motion in pipes including conservation of mass, momentum, and energy. We will then analyze flow in pipe systems subject to energy losses, and cover models for describing (major) frictional losses and (minor) losses due to changes in the geometry of flow. Finally, we will discuss how systems consisting of interconnected pipes and reservoirs can be analyzed and designed using the energy equation along with appropriate major and minor loss models.

General characteristics of pipe flow

First, let's distinguish between *pressurized pipe flow* and *open channel flow*. Both types of flow are governed by the same basic physical laws, including conservation of mass and momentum. However, there are important differences between the two that will dictate how we analyze these systems mathematically.

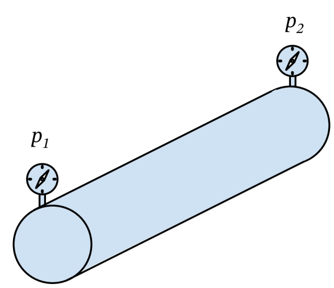


Figure 3: Pressurized pipe flow.

Pressurized pipe flow occurs in closed conduits that are completely filled with fluid. Because the conduit is completely full and the fluid is nearly incompressible, it can be assumed that the flow rate into a given control volume is equal to the flow rate out. Flow is generally driven by pressure gradients through the pipe, with water moving from high pressure zones to low pressure zones.

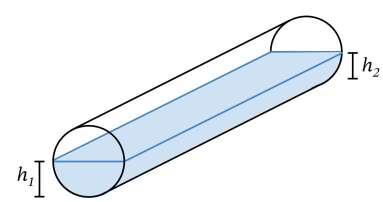


Figure 4: Open channel flow.

Open channel flow occurs in open channels and conduits that do not flow completely full. The flow rates into and out of any given control volume may not necessarily be equal because transient accumulation of water may occur (i.e. the control volume may fill or empty over time). The flow of water may be driven by gravity, pressure gradients, or the inertia of the water itself.

Basics of flow in pipes

The *discharge* or *flow rate* through a pipe is the volume of fluid that passes through a given cross section per unit time. You can think of the discharge as the product of the flow velocity and the cross sectional area. In general, the velocity of flow in a pipe will vary over its cross section, with the largest velocities in the center and the lowest velocities at the pipe wall, as shown in Figure 5.

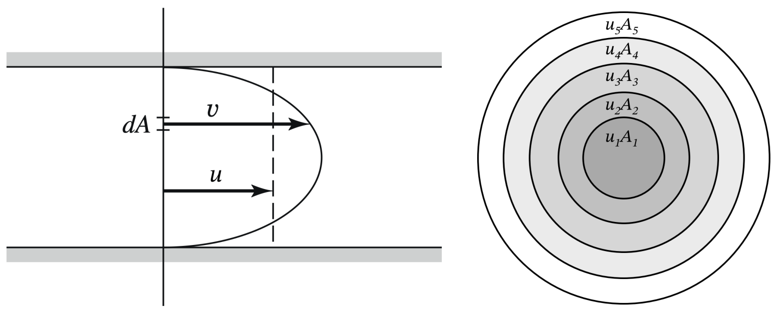


Figure 5: Left: velocity distribution in pipe flow (profile view) [2]. Right: velocity distribution in pipe flow (cross-sectional view).

If we know the velocity of each differential area that comprises the pipe's cross section, we can compute the total discharge by multiplying each differential area by its corresponding velocity and adding up the results. Let v be the flow velocity through differential area dA. The rate of flow through a differential area dA is thus $v \cdot dA$. The total discharge, Q, through the pipe is thus obtained by integrating over the entire cross section:

$$Q = \int_A v \cdot dA \tag{1}$$

EXAMPLE 1.1

Problem: A circular pipe with a radius of $R = 1$ $[m]$ carries water with a velocity distribution characterized by the parabolic function:

$$v = 1 - r^2 \quad [m/s] \tag{2}$$

Where r is the radial distance from the center of the pipe. What is the total discharge through the pipe?

Solution: The discharge is obtained by integration in polar coordinates:

$$Q = \int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr \, d\theta \tag{3}$$
$$= \int_0^{2\pi} \int_0^1 (r - r^3) \, dr \, d\theta \tag{4}$$
$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta \tag{5}$$
$$= \frac{1}{4} \int_0^{2\pi} d\theta \tag{6}$$
$$= \boxed{\frac{\pi}{2} \quad [m^3/s]} \tag{7}$$

Mean flow velocity

If we define the *mean velocity* of flow u, then the discharge is equal to the product of the mean velocity and the cross-sectional area A:

$$Q = uA \tag{8}$$

Consequently, the mean velocity of flow u is defined as:

$$u = \frac{Q}{A} \tag{9}$$

Where Q is the flow rate and A is the cross sectional area of flow.

EXAMPLE 1.2

Problem: Water flows through a pipe of diameter $D = 5$ $[ft]$ at a flow rate of $Q = 25$ $[ft^3/s]$. Find the mean velocity of flow, u.

Solution: Using the definition of mean flow velocity:

$$u = \frac{Q}{A} = Q\frac{4}{\pi D^2} = \frac{4Q}{\pi D^2} = \boxed{1.27 \ [ft/s]} \tag{10}$$

EXAMPLE 1.3

Problem: Water flows through a nozzle with a diameter $D_1 = 5$ $[ft]$ at the upstream end and $D_2 = 1$ $[ft]$ at the downstream end. The flow rate at the upstream end is $Q_1 = 25$ $[ft^3/s]$. Find the mean velocity of flow, u, at the downstream end.

Solution: We know that at steady state, the discharge at the upstream end is equal to the discharge at the downstream end:

$$Q_1 = Q_2 \tag{11}$$

From the definition of the mean velocity, we have that:

$$u_2 = \frac{Q_2}{A_2} = \frac{4Q_2}{\pi D^2} = \boxed{31.8\ [ft/s]} \tag{12}$$

Conservation laws

This section will cover the basic conservation laws that govern fluid flow, including the conservation of mass and the conservation of momentum. We will first derive the equations for these two conservation laws for a control volume consisting of a section of pipe. Next, using these two basic governing equations of fluid motion we will derive the equation for conservation of energy and show how this equation can be used to solve hydraulic design problems under steady-state conditions.

Conservation of mass

Considering fluid flow through a control volume, conservation of mass states that the *time rate of change of mass within the control volume is equal to the mass flow rate in minus the mass flow rate out*. Let's consider a control volume under consisting of a discrete length of pipe Δx over a discrete time interval Δt:

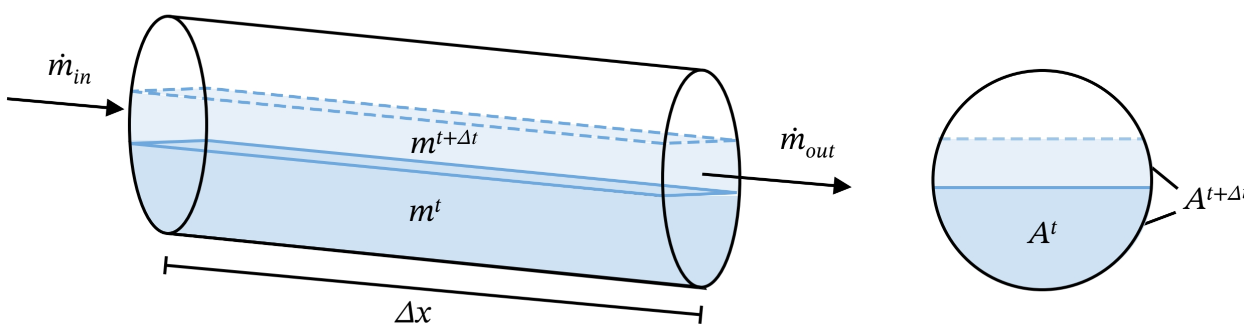


Figure 6: Continuity in a cylindrical pipe.

By conservation of mass, the rate of change in fluid mass inside the pipe is equal to the mass flow rate in minus the mass flow rate out:

$$\frac{m^{t+\Delta t} - m^t}{\Delta t} = \dot{m}_{in} - \dot{m}_{out} \tag{13}$$

Where m is the mass in the control volume, and $\dot{m}$ represents mass flow rate. Because mass is equal to density times volume ($m = \rho \forall$), and mass flow rate is equal to density times volumetric flow rate ($\dot{m} = \rho Q$), we have that:

$$\rho \frac{\forall^{t+\Delta t} - \forall^{t}}{\Delta t} = \rho Q_{in} - \rho Q_{out} \tag{14}$$

Assuming that the density of the fluid does not change within the control volume, we may divide both sides by the density ρ:

$$\frac{\forall^{t+\Delta t} - \forall^{t}}{\Delta t} = Q_{in} - Q_{out} \tag{15}$$

The volume of water is equal to the cross-sectional area times the length of the control volume ($\forall = \Delta x A$). Thus, the left-hand side can be simplified by factoring out Δx:

$$\Delta x \frac{A^{t+\Delta t} - A^{t}}{\Delta t} = Q_{in} - Q_{out} \tag{16}$$

Dividing both sides by the length of the control volume Δx:

$$\frac{A^{t+\Delta t} - A^{t}}{\Delta t} = \frac{Q_{in} - Q_{out}}{\Delta x} \tag{17}$$

Bringing the discharge terms to the left-hand side:

$$\frac{A^{t+\Delta t} - A^{t}}{\Delta t} + \frac{Q_{out} - Q_{in}}{\Delta x} = 0 \tag{18}$$

Taking the limit as $x \rightarrow 0$ and $t \rightarrow 0$, we arrive at the standard differential form of the continuity equation:

Conservation of mass equation in differential form

$$\boxed{\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0} \tag{19}$$

The above expression is the differential form of *conservation of mass* (otherwise known as *continuity*) for fluid flow. This equation is valid for both pipes and open channels in one spatial dimension. The first term ($\partial A / \partial t$) describes the time rate of change of volume (and thus mass) within the control volume. The second term ($\partial Q / \partial x$) describes the volumetric flux (and thus mass flux) through the control volume.

Conservation of mass for pressurized pipe flow

For pressurized pipe flow, the conservation of mass equation can be simplified further by assuming that the pipe is completely full and the fluid is incompressible.

In this case, the change in mass within the control volume over time is zero because the pipe is full ($\partial A / \partial t = 0$), and hence the flow

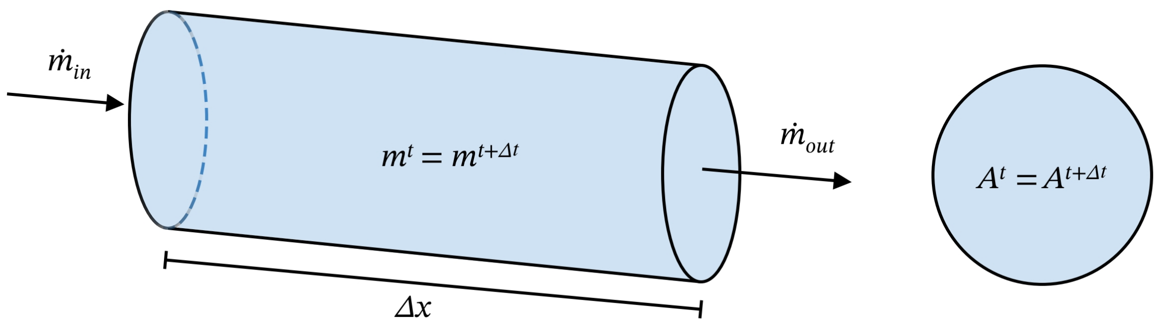


Figure 7: Continuity in a pressurized cylindrical pipe flowing completely full.

into the control volume is equal to the flow out of the control volume ($\partial Q/\partial x = 0$):

Conservation of mass for pressurized pipe flow

$$\boxed{\frac{\partial A}{\partial t} = \frac{\partial Q}{\partial x} = 0} \tag{20}$$

Conservation of momentum

For a closed system (like a control volume of fluid in a pipe), the *conservation of momentum* states that momentum remains constant in the absence of external forces. For a fluid system in which external forces are present, Newton's second law of motion extends this principle by relating the change in momentum to the applied external force. We know that for a point mass, Newton's second law states that the sum of forces acting on a body is equal to the mass of the body times its acceleration:

$$\sum F = ma \tag{21}$$

However, Newton's second law states more generally that *a body's time rate of change of momentum is equal to the sum of forces acting on the body*.

$$\frac{d(mu)}{dt} = \sum F \tag{22}$$

Where m is the mass and u is the velocity. We may thus determine the equations of motion for a fluid body by applying this principle over a fixed control volume.

Force balance

First, let's consider the right-hand side of Newton's second law, corresponding to the balance of forces. Figure 8 shows the forces acting

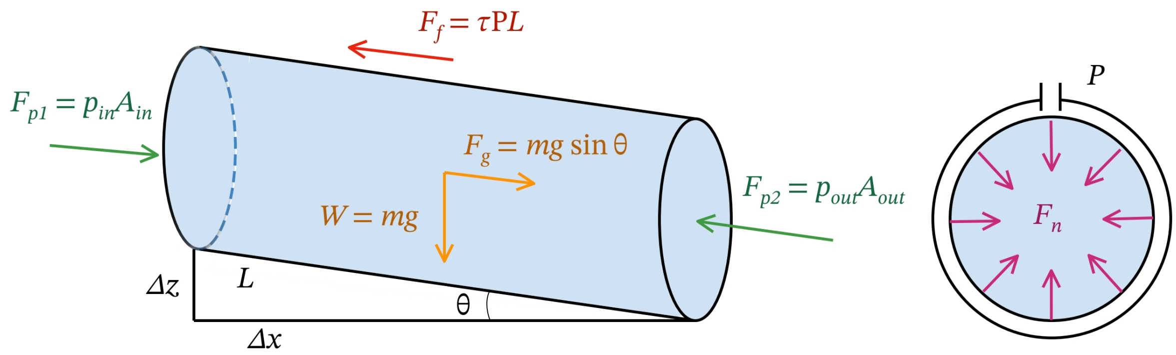


Figure 8: Force balance in a cylindrical pipe. Left: profile view. Right: Cross sectional view.

on the fluid in a pressurized pipe section. The forces acting on the fluid body include:

Pressure forces: The forces induced by the water pressure on the upstream and downstream boundaries of the control volume. The net pressure force in the longitudinal direction is given by:

$$F_p = p_{in}A_{in} - p_{out}A_{out} \tag{23}$$

Where p_{in} and p_{out} are the pressures at the upstream and downstream boundaries, respectively.

Gravitational forces: The component of the gravitational force in the longitudinal direction. The gravitational force in the longitudinal direction is given by:

$$F_g = mg\sin\theta = g\rho A\Delta x \cdot \sin\theta \tag{24}$$

Frictional forces: The force induced by friction at the conduit walls and within the fluid due to viscous interactions. The frictional force in the longitudinal direction is given by:

$$F_f = -\tau PL \approx -\tau P\Delta x \tag{25}$$

Where P is the wetted perimeter, and τ is the mean boundary shear stress.

Momentum

From the Reynolds transport theorem, the rate of change of momentum for a fluid body is defined as the sum of (1) the rate of change of momentum inside the fluid body and (2) the net flux of momentum through the boundaries of the control volume.

$$\begin{aligned} &\text{[Net rate of change of momentum] =} \\ &\text{[Rate of change of momentum inside the control volume]} \\ &\text{+ [Momentum flux out of control volume]} \\ &\text{- [Momentum flux into control volume]} \end{aligned} \tag{26}$$

We can thus approximate the change in momentum over a discrete time interval Δt and a discrete length of conduit Δx:

$$\frac{d(mu)}{dt} \approx \left[\frac{m^{t+\Delta t}u^{t+\Delta t} - m^t u^t}{\Delta t}\right] + \left[\dot{m}_{out}u_{out} - \dot{m}_{in}u_{in}\right] \tag{27}$$

$$= \rho\Delta x\left[\frac{A^{t+\Delta t}u^{t+\Delta t} - A^t u^t}{\Delta t}\right] + \rho\left[Q_{out}u_{out} - Q_{in}u_{in}\right] \tag{28}$$

$$= \rho\Delta x\left[\frac{Q^{t+\Delta t} - Q^t}{\Delta t}\right] + \rho\left[Q_{out}u_{out} - Q_{in}u_{in}\right] \tag{29}$$

Full conservation of momentum equation

Putting both sides of the equation together, we can write the following discrete approximation of the momentum equation:

$$\begin{aligned} &\rho\Delta x\left[\frac{Q^{t+\Delta t} - Q^t}{\Delta t}\right] + \rho\left[Q_{out}u_{out} - Q_{in}u_{in}\right] \\ &= \left[p_{in}A_{in} - p_{out}A_{out}\right] + g\rho\left[A\Delta x \cdot \sin\theta\right] + \left[-\tau P\Delta x\right] \end{aligned} \tag{30}$$

Dividing by Δx and ρ:

$$\begin{aligned} &\left[\frac{Q^{t+\Delta t} - Q^t}{\Delta t}\right] + \left[\frac{Q_{out}u_{out} - Q_{in}u_{in}}{\Delta x}\right] \\ &= g\left[\frac{p_{in}A_{in} - p_{out}A_{out}}{g\rho\Delta x}\right] + \left[gA \cdot \sin\theta\right] + \left[\frac{-\tau P}{\rho}\right] \end{aligned} \tag{31}$$

Using the small angle approximation, the $\sin\theta$ term is approximately equal to the bottom slope S_0:

$$S_0 = -\frac{\Delta z}{\Delta x} \approx \sin\theta \tag{32}$$

And similarly, we can define a friction slope, S_f, which represents the rate of head loss due to friction per unit length:

$$S_f = \frac{\tau P}{g\rho A} = \frac{\tau}{g\rho R} \tag{33}$$

Thus, we have:

$$\begin{aligned} &\left[\frac{Q^{t+\Delta t} - Q^t}{\Delta t}\right] + \left[\frac{Q_{out}u_{out} - Q_{in}u_{in}}{\Delta x}\right] \\ &= g\left[\frac{p_{in}A_{in} - p_{out}A_{out}}{g\rho\Delta x}\right] + \left[gA \cdot S_0\right] + \left[gA \cdot S_f\right] \end{aligned} \tag{34}$$

Taking the limit as $\Delta t \to 0$ and $\Delta x \to 0$, and then rearranging, we arrive at the differential form of the conservation of momentum:

Conservation of momentum for fluid flow in differential form

$$\boxed{\frac{\partial Q}{\partial t} + \frac{\partial (Qu)}{\partial x} + gA\left(\frac{1}{\gamma}\frac{\partial p}{\partial x} - S_0 + S_f\right) = 0} \tag{35}$$

Conservation of mass for pressurized pipe flow

For pressurized pipe flow, the conservation of momentum may be simplified further. Note that by the product rule, the momentum flux term can be partitioned:

$$\frac{\partial (Qu)}{\partial x} = Q\frac{\partial u}{\partial x} + u\frac{\partial Q}{\partial x} \tag{36}$$

Similarly, by the product rule, the time rate of change of momentum can be partitioned:

$$\frac{\partial Q}{\partial t} = \frac{\partial (Au)}{\partial t} = A\frac{\partial u}{\partial t} + u\frac{\partial A}{\partial t} \tag{37}$$

From the conservation of mass equation for pressurized pipe flow, $\partial Q/\partial x = 0$ and $\partial A/\partial t = 0$, and thus the conservation of momentum equation may be simplified as follows:

$$A\frac{\partial u}{\partial t} + Q\frac{\partial u}{\partial x} + gA\left(\frac{1}{\gamma}\frac{\partial p}{\partial x} - S_0 + S_f\right) = 0 \tag{38}$$

Dividing by the cross-sectional area yields the form of the momentum equation for pressurized pipe flow:

Conservation of momentum for pressurized pipe flow

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\left(\frac{1}{\gamma}\frac{\partial p}{\partial x} - S_0 + S_f\right) = 0} \tag{39}$$

EXAMPLE 1.4

Problem: Consider a horizontal section of pipe that is $\Delta x = 20$ [ft] long and $D = 6$ [in] in diameter. Water moves through the pipe with a constant flow rate of $Q = 20$ [ft^3/s]. The pressure at the upstream end of the pipe section is $p_1 = 60$ [psi] and at the downstream end of the pipe is $p_2 = 55$ [psi].

1. What is the force exerted by friction in the pipe?

2. What is the energy loss due to friction?

Solution: Applying the conservation of momentum to the entire control volume, we have:

$$\sum F = \frac{d(mu)}{dt} \tag{40}$$

The sum of forces includes pressure, gravitational and frictional forces. The net rate of change of momentum is equal to the change in momentum within the control volume plus the net momentum flux through the control volume.

$$F_p + F_g - F_f = \rho \Delta x \frac{dQ}{dt} + \rho(Q_{out}u_{out} - Q_{in}u_{in}) \tag{41}$$

Because the system is at steady-state, the time rate of change of momentum is zero:

$$F_p + F_g - F_f = \cancelto{0}{\rho \Delta x \frac{dQ}{dt}} + \rho(Q_{out}u_{out} - Q_{in}u_{in}) \tag{42}$$

Now, note that by continuity, the flow rate in equals the flow rate out:

$$Q_{in} = Q_{out} = Q \tag{43}$$

Moreover, the cross-sectional area is constant, such that $A_{in} = A_{out} = A$. Thus, the velocities at the upstream and downstream end are equal:

$$u_{in} = \frac{Q}{A_{in}} = \frac{Q}{A} = \frac{Q}{A_{out}} = u_{out} \tag{44}$$

Thus, the net momentum flux through the pipe is zero:

$$\rho(Q_{out}u_{out} - Q_{in}u_{in}) = \rho(Qu - Qu) = 0 \tag{45}$$

Thus, the system is at static equilibrium and our sum of forces is equal to zero:

$$\sum F = 0 \tag{46}$$

$$F_p + F_g - F_f = 0 \tag{47}$$

$$(p_1 A_1 - p_2 A_2) + mg \sin\theta - F_f = 0 \tag{48}$$

Because the pipe is horizontal, $\theta = 0$ and the gravitational force component in the longitudinal direction is zero. Thus, the frictional force is equal to the net pressure force:

$$F_f = p_1 A_1 - p_2 A_2 \tag{49}$$

Note that $A_1 = A_2 = A$, thus:

$$F_f = \frac{\pi D^2}{4}(p_1 - p_2) \tag{50}$$

Substituting in values, we have:

$$F_f = \frac{\pi(6\ [in])^2}{4}(60\ [lb/in^2] - 55\ [lb/in^2]) \tag{51}$$

$$\boxed{F_f = 141\ [lb]} \tag{52}$$

To compute the energy loss due to friction, recall that work is equal to the integral of force over the distance applied. Thus we have that:

$$E_L = \int_0^{\Delta x} F_f dx = F_f \Delta x = 141\ [lb] \cdot 20\ [ft] \tag{53}$$

$$\boxed{E_L = 2,820\ [ft \cdot lb]} \tag{54}$$

The Saint-Venant Equations

Combining together conservation of mass and conservation of momentum, we arrive at the *Saint-Venant Equations* for fully unsteady flow in one-dimensional pipes or open channels:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{55}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial (Qu)}{\partial x} + gA\left(\frac{1}{\gamma}\frac{\partial p}{\partial x} - S_0 + S_f\right) = 0 \tag{56}$$

These coupled equations fully describe the dynamics of unsteady one-dimensional flow, and form the basis for computational fluid dynamics solvers like HEC-RAS and EPA-SWMM.

Each term in the equations relates to an element of the mass or momentum balance:

- $\partial A/\partial t$ represents the change in mass within the control volume
- $\partial Q/\partial x$ represents the mass flux through the control volume (flow in minus flow out)
- $\partial Q/\partial t$ represents the change in momentum within the control volume
- $\partial (Qu)/\partial x$ represents the momentum flux through the control volume
- $\partial p/\partial x$ represents the pressure force
- S_0 represents the gravitational force
- S_f represents the frictional force

Conservation of energy

You may recall the *energy equation* from your earlier fluid mechanics class. We can derive this equation from the momentum equation under two conditions:

1. Assume the flow is steady-state ($\partial/\partial t = 0$)
2. Integrate between an upstream point (1) and a downstream point (2).

Using the conservation of momentum equation for pressurized fluid flow, we have:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\left(\frac{1}{\gamma}\frac{\partial p}{\partial x} - S_0 + S_f\right) = 0 \tag{57}$$

Assuming steady state conditions (all time derivatives are zero):

$$u\frac{\partial u}{\partial x} + g\left(\frac{1}{\gamma}\frac{\partial p}{\partial x} - S_0 + S_f\right) = 0 \tag{58}$$

Dividing both sides by g, we get:

$$\frac{u}{g}\frac{\partial u}{\partial x} + \frac{1}{\gamma}\frac{\partial p}{\partial x} - S_0 + S_f = 0 \tag{59}$$

Note that S_0 is the bottom slope, and is thus equal to $-\frac{\partial z}{\partial x}$. Similarly, S_f is the slope of the energy grade line, and is equal to the energy loss due to friction per unit length. We can write this energy loss per unit length in differential form as $\frac{\partial e}{\partial x}$. Thus we have:

$$\frac{u}{g}\frac{\partial u}{\partial x} + \frac{1}{\gamma}\frac{\partial p}{\partial x} + \frac{\partial z}{\partial x} + \frac{\partial e}{\partial x} = 0 \tag{60}$$

Now, let's integrate both sides from the upstream point x_1 to the downstream point x_2:

$$\int_{x_1}^{x_2}\left(\frac{u}{g}\frac{\partial u}{\partial x} + \frac{1}{\gamma}\frac{\partial p}{\partial x} + \frac{\partial z}{\partial x} + \frac{\partial e}{\partial x}\right)\partial x = 0 \tag{61}$$

Expanding, we recognize that the increments of x cancel:

$$\frac{1}{g}\int_{x_1}^{x_2} u\frac{\partial u}{\cancel{\partial x}}\cancel{\partial x} + \int_{x_1}^{x_2}\frac{\partial p}{\cancel{\partial x}}\cancel{\partial x} + \int_{x_1}^{x_2}\frac{\partial z}{\cancel{\partial x}}\cancel{\partial x} + \int_{x_1}^{x_2}\frac{\partial e}{\cancel{\partial x}}\cancel{\partial x} = 0 \tag{62}$$

And thus, we can adjust the limits of integration to obtain:

$$\frac{1}{g}\int_{u_1}^{u_2} u\partial u + \frac{1}{\gamma}\int_{p_1}^{p_2}\partial p + \int_{z_1}^{z_2}\partial z + \int_{e_1}^{e_2}\partial e = 0 \tag{63}$$

Integrating, we get:

$$\left[\frac{u_2^2}{2g} - \frac{u_1^2}{2g}\right] + \left[\frac{p_2}{\gamma} - \frac{p_1}{\gamma}\right] + [z_2 - z_1] + h_L = 0 \tag{64}$$

Where h_L is the energy (head) loss due to friction. Then we have the *energy equation*:

Energy equation for a fluid

$$\boxed{\frac{u_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{u_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + h_L} \quad (65)$$

The above equation is also sometimes called *Bernoulli's equation*, but it is fundamentally just an energy balance.

Note that if the distance between points x_1 and x_2 is Δx, then the head loss due to friction is $h_L = \Delta x \cdot S_f$. Similarly, the difference in bed slope elevation $z_1 - z_2 = \Delta x \cdot S_0$. The head loss term in this derivation is the head loss due to friction, but this term may in general also include other head losses (such as head losses due to expansions or contractions).

Example 1.5

Problem: An elevated water tank is being drained into an underground storage location through a 12-inch diameter pipe. The flow rate is 3200 gallons per minute (gpm) and the total head loss is 11.53 ft. Determine the water surface elevation in the tank.

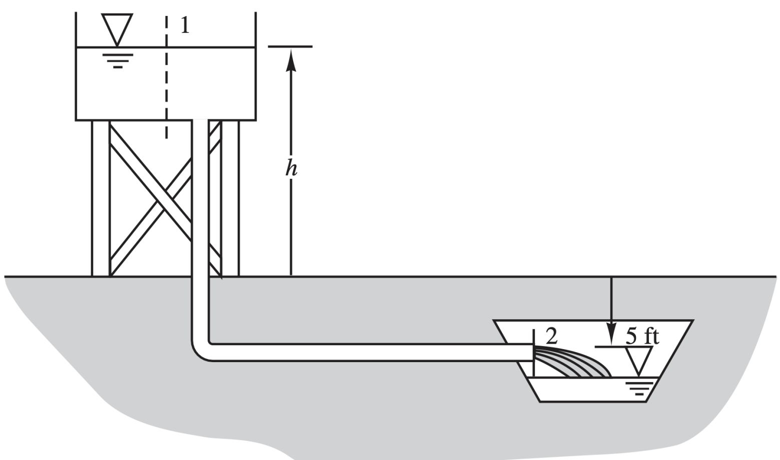


Figure 9: Elevated tank and underground storage [1].

Solution: Using the energy equation between section 1 at the reservoir and section 2 at the end of the pipe:

$$\frac{u_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{u_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + h_L \quad (66)$$

The water velocity in the reservoir is negligible such that $u_1 = 0$. The reservoir and the end of the pipe are both exposed to atmospheric pressure, such that:

$$p_1 = p_2 = 0 \tag{67}$$

The mean velocity is:

$$u = \frac{Q}{A} = \frac{Q}{\pi r^2} = \frac{3200 \text{ gpm}}{(0.5 \text{ ft})^2 \pi} \cdot \frac{1 \text{ ft}^3/\text{s}}{448.83 \text{ gpm}} = 9.08 \text{ ft/s} \tag{68}$$

Thus:

$$h = h_1 = \frac{u_2^2}{2g} + h_2 + h_L = \frac{(9.08)^2}{2(32.2)} - 5 + 11.53 = \boxed{7.81 \text{ ft}} \tag{69}$$

A note on energy versus head

Our energy equation is expressed in terms of units of length (e.g. m) rather than units of energy (e.g. Nm). How did this happen? Remember that in our derivation, we divided the energy by four different quantities:

1. The density of the fluid ρ in units of $\left[\frac{M}{L^3}\right]$ (Eq. 34).
2. The length of our control volume Δx in units of $[L]$ (Eq. 34).
3. The cross-sectional area of our control volume A in units of $[L^2]$ (Eq. 39).
4. The acceleration due to gravity g in units of $\left[\frac{L}{T^2}\right]$ (Eq. 59)

Energy is expressed in units of $[ML^2/T^2]$. Thus, applying these operations results in units of $[L]$:

$$\left[\frac{ML^2}{T^2}\right] \cdot \left[\frac{L^3}{M}\right] \cdot \left[\frac{1}{L}\right] \cdot \left[\frac{1}{L^2}\right] \cdot \left[\frac{T^2}{L}\right] = \left[L\right] \tag{70}$$

In hydraulics, it is often convenient to express energy in terms of head. You may think of the head as representing the *energy per unit weight of water*.

Types of head

For convenience, head is often classified into the following types:

Total head: the total head corresponding to the total energy.

$$H = \frac{p}{\gamma} + z + \frac{u^2}{2g} \tag{71}$$

Elevation head: the head corresponding to the potential energy associated with a given elevation above some datum.

$$H_e = z \tag{72}$$

Pressure head: the head corresponding to the energy associated with the pressure of the fluid.

$$H_p = \frac{p}{\gamma} \tag{73}$$

Velocity head: the head corresponding to the kinetic energy of the fluid.

$$H_v = \frac{u^2}{2g} \tag{74}$$

Piezometric head: the sum of the pressure head and the elevation head (equal to the elevation of the water surface for open channel flow).

$$H_h = \frac{p}{\gamma} + z \tag{75}$$

Hydraulic and energy grade lines

The energy along a conduit or open channel can be visualized using the hydraulic grade line and the energy grade line:

Energy Grade Line (EGL): The EGL is the line that connects all total energy points along a conduit or open channel. The slope of the EGL indicates the rate at which energy is lost (due to friction, contractions, expansions, etc.). The EGL is defined as:

$$EGL(x) = z(x) + \frac{p(x)}{\gamma} + \frac{u^2(x)}{2g} \tag{76}$$

Hydraulic Grade Line (HGL): The HGL is the line indicating the piezometric head (i.e. the sum of the elevation head and the pressure head). For an open channel, the HGL indicates the absolute elevation of the free water surface. The HGL is defined as:

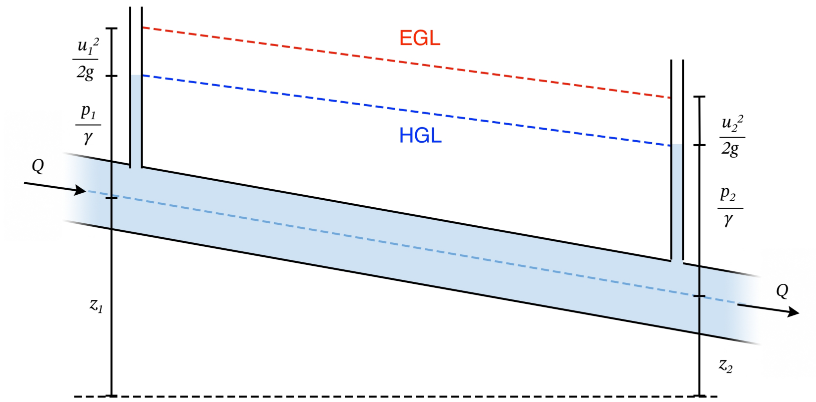


Figure 10: Visualization of hydraulic grade line (HGL) and energy grade line (EGL) in a pressurized pipe.

$$HGL(x) = z(x) + \frac{p(x)}{\gamma} \tag{77}$$

Typically, the velocity head (kinetic energy) is small compared to the piezometric head (potential energy) and thus the EGL and HGL will generally be close together in magnitude. The EGL and HGL are parallel for a pipe section with uniform diameter. Note that by convention we define the water pressure in terms of gage pressure $p = p_{gauge}$, where the gage pressure is the absolute pressure minus the ambient atmospheric pressure:

$$p_{gauge} = p_{abs} - p_{atm} \tag{78}$$

By definition, the gage pressure is equal to zero ($p_{gauge} = 0$) at the interface between the water surface and the atmosphere.

Frictional losses

A variety of models have been developed to determine head losses due to friction, including the Darcy-Weisbach, Chezy, Manning, and Hazen-Williams formulas. In this section, we will focus on the Darcy-Weisbach equation, which is a popular model for head losses in pressurized cylindrical pipes that applies to both laminar and turbulent flows.

Chezy equation

Recall that the friction slope is equal to the energy loss per unit length, and is defined as:

$$S_f = \frac{\tau P}{gA\rho} = \frac{\tau}{g\rho R} \tag{79}$$

Where R is the hydraulic radius, which is equal to the cross-sectional area of flow divided by the wetted perimeter:

$$R = \frac{A}{P} \tag{80}$$

Thus, the mean shear stress can be expressed as:

$$\tau = g\rho R S_f \tag{81}$$

From Newton's drag law, shear stress for fully turbulent flow can be expressed as a function of density, velocity, and a resistance coefficient C_f:

$$\tau = C_f \rho \frac{u^2}{2} \tag{82}$$

Equating the two expressions for the shear stress:

$$g\rho R S_f = C_f \rho \frac{u^2}{2} \tag{83}$$

Rearranging for the friction slope yields:

$$S_f = C_f \frac{u^2}{2gR} \tag{84}$$

Recall that the frictional head losses are equal to $h_L = S_f \Delta x$, and thus:

Chezy equation (in terms of head loss)

$$\boxed{h_L = C_f \frac{\Delta x}{R} \frac{u^2}{2g}} \tag{85}$$

Alternatively, solving for the velocity gives:

$$u = \sqrt{\frac{2g}{C_f}} \sqrt{R S_f} \tag{86}$$

Let's define $C = \sqrt{2g/C_f}$. Then the equation above can be simplified to the well-known *Chezy equation*:

$$u = C\sqrt{R S_f} \tag{87}$$

Multiplying by the cross-sectional area, we can express in terms of discharge:

Chezy equation (in terms of discharge)

$$\boxed{Q = CA\sqrt{R S_f}} \tag{88}$$

Where C is referred to as the *Chezy coefficient*. The other frictional loss formulas (Chezy-Manning, Darcy-Weisbach, Hazen-Williams) can all be thought of as different ways of specifying this coefficient.

Darcy-Weisbach equation

The Darcy-Weisbach equation is a special case of the Chezy equation for flow in a circular pipe. Starting with the Chezy model of head loss from equation (85), we have:

$$h_L = C_f \frac{\Delta x}{R} \frac{u^2}{2g} \tag{89}$$

First, note that the hydraulic radius $R = A/P$:

$$h_L = C_f \frac{\Delta x P}{A} \frac{u^2}{2g} \tag{90}$$

Now note that for a circular pipe $A = \pi D^2/4$ and $P = \pi D$:

$$h_L = C_f \frac{4\pi \Delta x D}{\pi D^2} \frac{u^2}{2g} = 4C_f \frac{\Delta x}{D} \frac{u^2}{2g} \tag{91}$$

The first term $4C_f$ may be rewritten as a single friction factor f. Thus, we arrive at the standard form of the Darcy-Weisbach equation:

Darcy-Weisbach equation for frictional head loss

$$\boxed{h_L = f \frac{\Delta x}{D} \frac{u^2}{2g}} \tag{92}$$

The friction factor f depends on whether the flow is turbulent or laminar. To understand frictional energy losses further, we must understand the differences between these two types of flow.

Laminar and turbulent flow

Fluid flow can be divided into turbulent and laminar flow:

Turbulent flow is characterized by eddies of varying sizes within the flow that create a mixing action. Fluid particles follow irregular and erratic paths. Mixing together of slow and fast particles creates a more even logarithmic velocity profile in pipes.

Laminar flow is characterized by smooth streamlines. Fluid particles move in definite paths with little mixing. Flow appears as sliding laminations of infinitesimal thickness relative to the adjacent layers. For pipe flow, there is a distinct parabolic cross sectional velocity profile, with faster particles traveling at the center of the pipe.

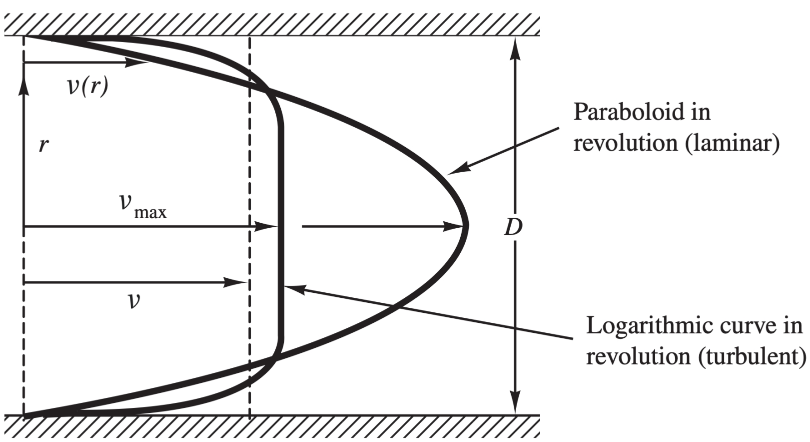


Figure 11: Laminar versus turbulent velocity profiles [2].

Reynolds' experiment

The transition between laminar and turbulent flow was explored experimentally by British engineer Osborne Reynolds (1842–1912). Reynolds' original experiment introduced dye into a stream of water. The velocity of water was controlled by a valve at the outlet. As the velocity of the water increased, the smooth streamline of dye associated with laminar flow broke up into a turbulent mixture.

Reynolds' experiment determined that the transition from laminar to turbulent flow depended not only on the velocity of the fluid, but also on the diameter of the pipe and the viscosity of the fluid.

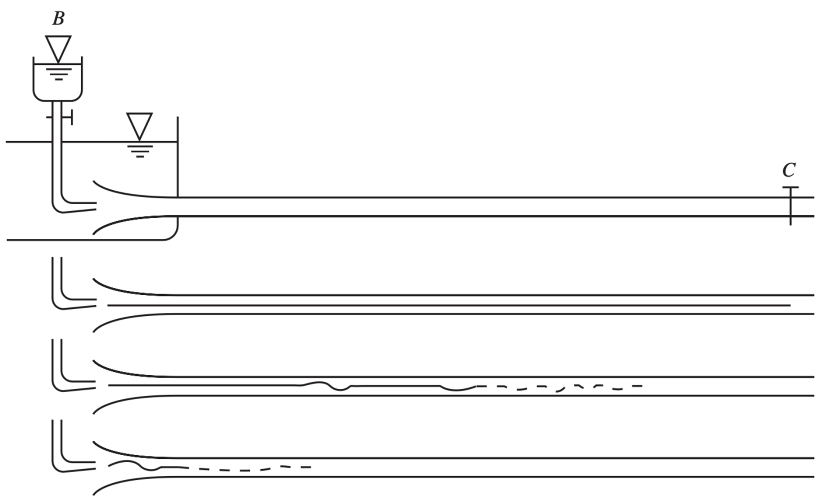


Figure 12: Reynolds' experiment showing laminar (top), transitional (middle), and turbulent (bottom) flow [1].

Reynolds Number

The *Reynolds number* can be used to determine whether flow is turbulent or laminar. For a circular pipe, the *Reynolds Number* (R_e) is given as:

Reynolds number (in terms of kinematic viscosity)

$$\boxed{R_e = \frac{Du\rho}{\mu}} \tag{93}$$

Where D is the diameter of the pipe, u is the mean velocity of flow, and μ is the dynamic viscosity of the fluid. The dynamic viscosity μ is defined as the ratio of shear stress to the velocity gradient with respect to the distance from the center of the pipe y:

$$\mu = \frac{\tau}{\partial u / \partial y} \tag{94}$$

The kinematic viscosity is defined as the dynamic viscosity divided by the density of the fluid:

$$\nu = \frac{\mu}{\rho} \tag{95}$$

Thus the Reynolds number can also be defined as:

Reynolds number (in terms of kinematic viscosity)

$$\boxed{R_e = \frac{Du}{\nu}} \tag{96}$$

Thresholds for laminar vs. turbulent flow

The Reynolds number determines the threshold between laminar and turbulent flow. Based on experiments, for flow in circular pipes the transition from laminar to turbulent occurs from $R_e \approx 2000 - 4000$.

- *Laminar flow:* $R_e \leq 2000$
- *Transitional flow:* $2000 < R_e \leq 4000$
- *Turbulent flow:* $R_e > 4000$

EXAMPLE 1.7

Problem: A 40mm diameter circular pipe carries water at 20.2° C. Calculate the largest flow rate for which laminar flow can be expected.

Solution: At normal conditions, the dynamic viscosity is:

$$\mu = 1.0 \times 10^{-3} \left[\frac{Ns}{m^2}\right] \quad (97)$$

The diameter of the pipe is $D = 0.04$ [m]. Taking $R_e = 2000$ as the conservative upper limit for laminar flow:

$$R_e = \frac{Du\rho}{\mu} \quad (98)$$

$$u = \frac{R_e\mu}{D\rho} = \frac{(2000)(1 \times 10^{-3}\ [kg \cdot m/s])}{(0.04\ [m])(1000\ [kg/m^3])} = 0.05\ [m/s] \quad (99)$$

The flow rate is thus:

$$Q = uA = (0.05\ [m/s])(\frac{\pi}{4}(0.04\ [m])^2) = \boxed{6.28 \times 10^{-5} \left[\frac{m^3}{s}\right]} \quad (100)$$

Friction losses in laminar flow

Recall that from the Darcy-Weisbach equation, the head loss due to friction in a circular pipe is:

$$h_L = f\frac{\Delta x}{D}\frac{u^2}{2g} \quad (101)$$

For laminar flow, it can be shown that the friction factor f has the simple closed-form solution (see [1], §3.5.1):

Darcy-Weisbach friction factor for laminar flow

$$\boxed{f = \frac{64}{R_e}} \quad (102)$$

Note that the Reynolds number only depends on the velocity and viscosity of the fluid and the pipe diameter. Thus, *for laminar flow the friction losses are independent of the roughness of the pipe*.

Friction losses in turbulent flow

For turbulent flow, the friction factor is commonly approximated by the implicit *Colebrook-White equation*:

Colebrook-White equation for friction factor f

$$\boxed{\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{e/D}{3.7} + \frac{2.51}{R_e\sqrt{f}}\right)} \tag{103}$$

Note that this equation is implicit, and generally must be solved either numerically or using graphical techniques. For the case of turbulent flow, the friction losses depend on the viscosity of the fluid, the pipe diameter, and the roughness of the pipe.

Numerical solution to Colebrook-White equation via Excel

Implicit equations like the Colebrook-White equation can be solved using numerical methods for root-finding. First, we subtract the right-hand side of the equation from both sides, such that the left-hand side equates to zero:

$$\frac{1}{\sqrt{f}} + 2\log_{10}\left(\frac{e/D}{3.7} + \frac{2.51}{R_e\sqrt{f}}\right) = 0 \tag{104}$$

This equation can be solved numerically using a root-finding algorithm such as bisection. Microsoft Excel offers functionality for solving implicit and nonlinear equations. To use this functionality, first ensure that the *Solver Add-in* for Excel is enabled:

- Navigate to `File -> Options -> Solver`
- Check the tickbox to enable the solver extension.

Next, set up the spreadsheet to solve the implicit equation:

1. Define input parameters
2. Define the *unknown variable*: x, and provide an initial guess of the unknown variable: x_0
3. Define an *objective function* of the form: $F(x) = 0$
4. Call the solution function

The following example illustrates the use of Excel's solver to determine the Colebrook-White friction factor.

EXAMPLE 1.8

Problem: Consider a 4 [m] diameter riveted-steel pipe in its best condition. The pipe carries a discharge of $Q = 100\ [m^3/s]$ at normal conditions (water temperature of 20.2°C and ambient pressure of 760 [mm Hg]). What is the Colebrook-White friction factor?

Solution: The procedure for solving this problem with Excel is outlined as follows:

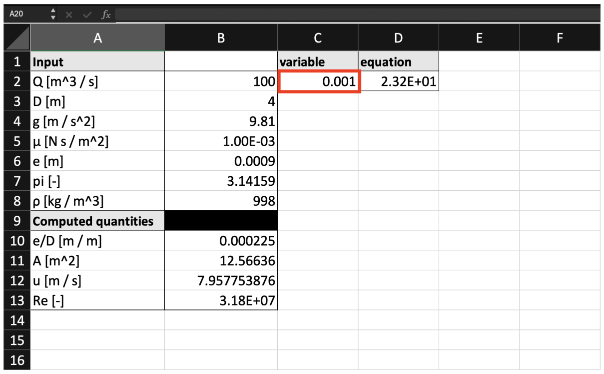
1a. Define input parameters: Enter constants and other parameters specific to the problem into a specified set of cells.

	A	B	C	D	E	F
1	Input		variable	equation		
2	Q [m^3 / s]	100	0.001	2.32E+01		
3	D [m]	4				
4	g [m / s^2]	9.81				
5	μ [N s / m^2]	1.00E-03				
6	e [m]	0.0009				
7	pi [-]	3.14159				
8	ρ [kg / m^3]	998				
9	Computed quantities					
10	e/D [m / m]	0.000225				
11	A [m^2]	12.56636				
12	u [m / s]	7.957753876				
13	Re [-]	3.18E+07				
14						
15						
16						

1b. Define computed quantities: Define computed quantities (such as the Reynolds number) for convenience.

	A	B	C	D	E	F
1	Input		variable	equation		
2	Q [m^3 / s]	100	0.001	2.32E+01		
3	D [m]	4				
4	g [m / s^2]	9.81				
5	μ [N s / m^2]	1.00E-03				
6	e [m]	0.0009				
7	pi [-]	3.14159				
8	ρ [kg / m^3]	998				
9	Computed quantities					
10	e/D [m / m]	0.000225				
11	A [m^2]	12.56636				
12	u [m / s]	7.957753876				
13	Re [-]	3.18E+07				
14						
15						
16						

2. Define unknown variable: Here, we specify a cell to hold the unknown variable (friction factor, f) and provide an initial guess of $f_0 = 0.001$.

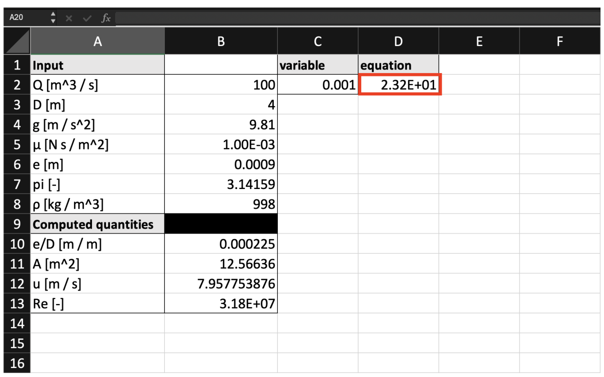


	A	B	C	D	E	F
1	Input		variable	equation		
2	Q [m^3 / s]	100	0.001	2.32E+01		
3	D [m]	4				
4	g [m / s^2]	9.81				
5	μ [N s / m^2]	1.00E-03				
6	e [m]	0.0009				
7	pi [-]	3.14159				
8	ρ [kg / m^3]	998				
9	Computed quantities					
10	e/D [m / m]	0.000225				
11	A [m^2]	12.56636				
12	u [m / s]	7.957753876				
13	Re [-]	3.18E+07				
14						
15						
16						

3. *Define objective function:* Here, we specify a cell to hold the objective function. In this case, our objective cell is given by the Colebrook-White equation as follows:

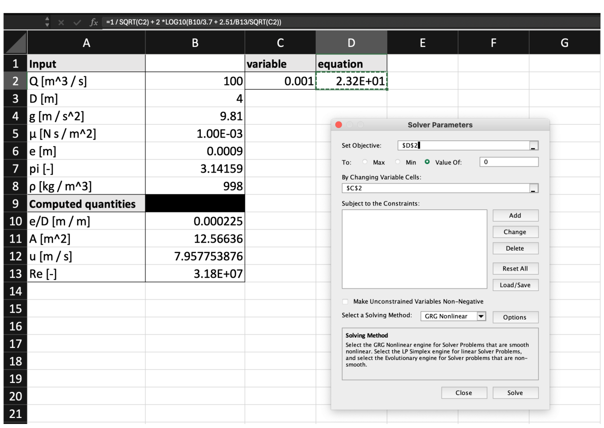
```
= 1 / SQRT(C2) + 2 * LOG10(B10 / 3.7 + 2.51 / B13 / SQRT(C2))
```

Note: to prevent the friction factor from becoming zero or negative, it may be necessary to include a safeguard: for instance, within the objective function, replace references to `C2` with `MAX(C2, 1e-5)`



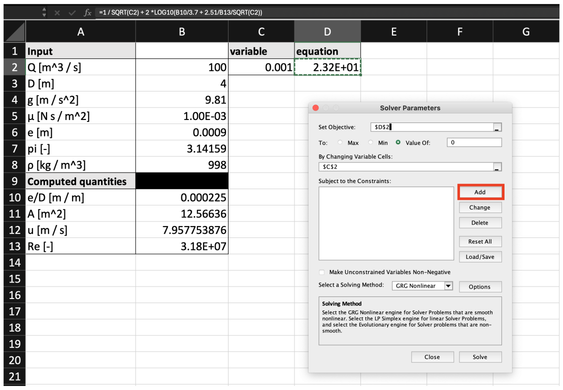
	A	B	C	D	E	F
1	Input		variable	equation		
2	Q [m^3 / s]	100	0.001	2.32E+01		
3	D [m]	4				
4	g [m / s^2]	9.81				
5	μ [N s / m^2]	1.00E-03				
6	e [m]	0.0009				
7	pi [-]	3.14159				
8	ρ [kg / m^3]	998				
9	Computed quantities					
10	e/D [m / m]	0.000225				
11	A [m^2]	12.56636				
12	u [m / s]	7.957753876				
13	Re [-]	3.18E+07				
14						
15						
16						

4a. *Set up solver:* Here, we click on the *Solver* icon under the *Data* tab in Excel. Note that the objective cell is `D2`, the variable cell is `C2`, and the solver is seeking to set the objective function to `Value of: 0`.



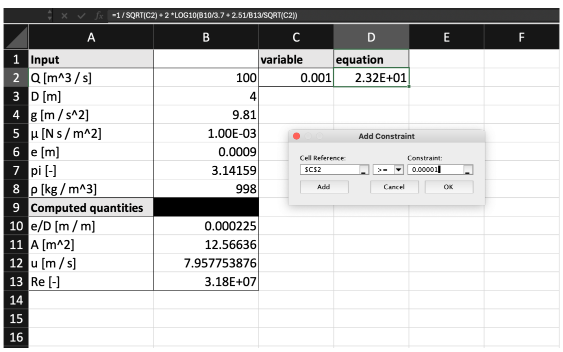
	A	B	C	D	E	F	G
1	Input		variable	equation			
2	Q [m^3 / s]	100	0.001	2.32E+01			
3	D [m]	4					
4	g [m / s^2]	9.81					
5	μ [N s / m^2]	1.00E-03					
6	e [m]	0.0009					
7	pi [-]	3.14159					
8	ρ [kg / m^3]	998					
9	Computed quantities						
10	e/D [m / m]	0.000225					
11	A [m^2]	12.56636					
12	u [m / s]	7.957753876					
13	Re [-]	3.18E+07					

4b. Set constraints: We will want to make sure that the friction factor is greater than zero. To do this, we can add a constraint using the *Add* button.

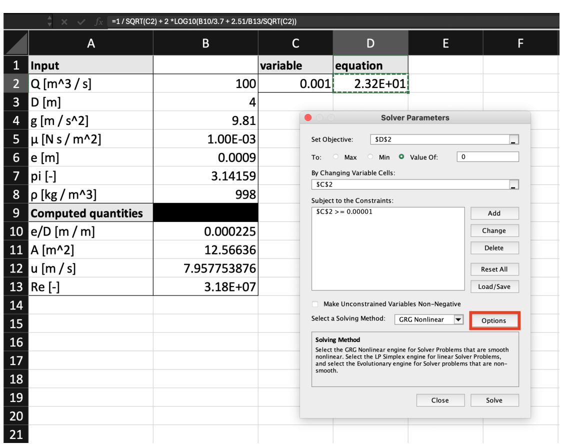


We will ensure that the friction factor remains greater than zero by setting cell C2 to be greater than a very small value (in this case, let `$C$2 >= 0.00001`).

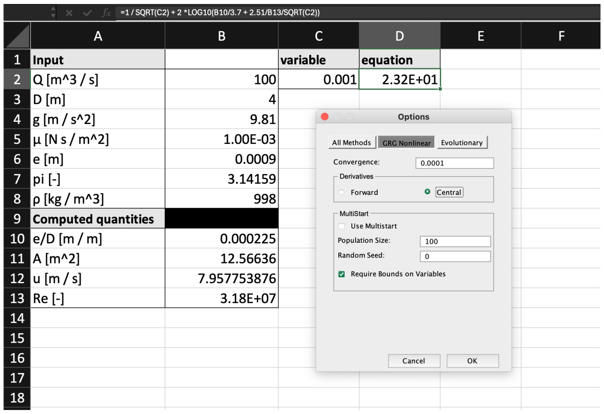
Note: on some computers, the solver may not respect this constraint, and safeguards must be placed on the value of `C2`, as described above.



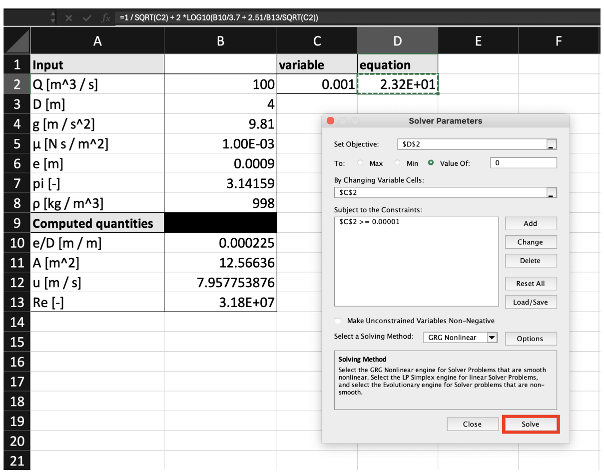
4c. Set solver options: To ensure better convergence, we will change the solver options under the *Options* button.



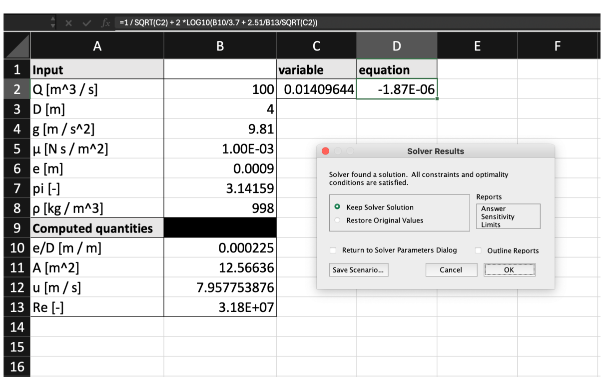
Under options for the *GRG Nonlinear* solver, check the radio dial to use *Central* derivatives. Then click *OK* to return to the main solver window.



4d. Call solver: After clicking *Solve* on the main solver window, the solver will populate the variable cell with the friction factor needed to set the value of the objective function to zero.



If the solver has converged, click *OK* to keep the solver solution.



In this case, the computed friction factor is:

$$\boxed{f = 0.014} \tag{105}$$

Solving a system of nonlinear equations using Excel

Sometimes we will need to solve a *system* of nonlinear equations. In the following example, we will determine both the friction factor and discharge in a pipe simultaneously by combining the energy equation, the Darcy-Weisbach equation for frictional losses, and the Colebrook-White equation for the friction factor:

EXAMPLE 1.9

Problem: Two sections A and B are 4.5 [km] apart along a 4 [m] diameter riveted-steel pipe in its best condition. A is 100 [m] higher than B. If water temperature is 20.2°C and the pressure heads measured at A and B are 8.3 [m] and 76.7 [m], respectively, what is the flow rate? Assume minor losses are negligible.

Solution: Using the energy equation between A and B:

$$\frac{u_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{u_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + h_L \qquad (106)$$

Rearranging:

$$h_L = \left(\frac{u_1^2}{2g} - \frac{u_2^2}{2g}\right) + \left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma}\right) + (z_1 - z_2) \qquad (107)$$

Applying continuity:

$$h_L = \left(\frac{Q^2}{2gA_1^2} - \frac{Q_2^2}{2gA^2}\right) + \left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma}\right) + (z_1 - z_2) \qquad (108)$$

Because the diameter of the pipe does not change, the first two terms are the same and thus cancel out. Similarly, from the problem description, we know $z_1 - z_2 = \Delta z$:

$$h_L = \left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma}\right) + \Delta z \qquad (109)$$

From Darcy-Weisbach, the head loss may be rewritten:

$$f\frac{\Delta x}{D}\frac{u^2}{2g} = \left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma}\right) + \Delta z \qquad (110)$$

Combining this expression with Colebrook-White, we have two equations with two unknowns (f, u):

$$f = \frac{2gD}{\Delta x u^2}\left[\left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma}\right) + \Delta z\right] \qquad (111)$$

$$0 = \frac{1}{\sqrt{f}} + 2\log_{10}\left(\frac{e/D}{3.7} + \frac{2.51}{R_e\sqrt{f}}\right) \qquad (112)$$

The general procedure for solving this system of equations in Excel is outlined as follows:

1a. Define input parameters: Enter constants and other parameters specific to the problem into a specified set of cells.

M15 | fx

	A	B	C	D	E	F
1	**Input**		**variable**	**equation**		
2	Δx [m]	4500	1	-8.31E+00		
3	D [m]	4				
4	g [m / s^2]	9.81				
5	μ [N s / m^2]	1.00E-03				
6	hL [m]	31.6				
7	e [m]	0.0009				
8	pi [-]	3.14159				
9	ρ [kg / m^3]	1000				
10	**Computed quantities**					
11	e/D [m / m]	0.000225				
12	u [m / s]	0.079577539				
13	Re [-]	3.18E+05				
14	f [-]	87.0267084				
15						

1b. Define computed quantities: Define computed quantities (such as the Reynolds number) for convenience.

B14 | fx =B6*2*B4*B3/(B2*B12^2)

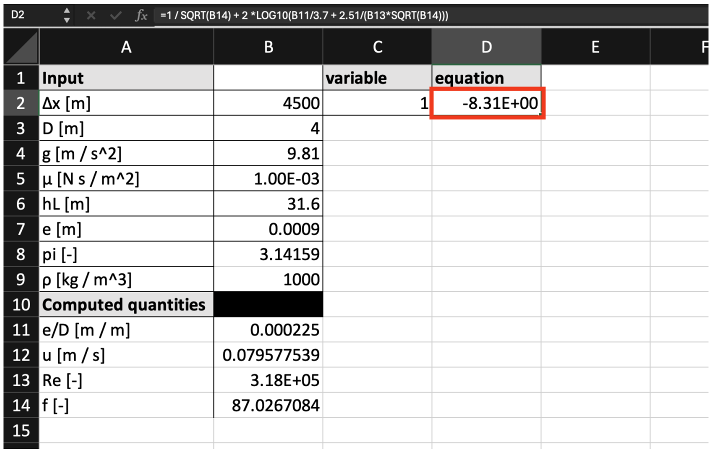
	A	B	C	D	E	F
1	**Input**		**variable**	**equation**		
2	Δx [m]	4500	1	-8.31E+00		
3	D [m]	4				
4	g [m / s^2]	9.81				
5	μ [N s / m^2]	1.00E-03				
6	hL [m]	31.6				
7	e [m]	0.0009				
8	pi [-]	3.14159				
9	ρ [kg / m^3]	1000				
10	**Computed quantities**					
11	e/D [m / m]	0.000225				
12	u [m / s]	0.079577539				
13	Re [-]	3.18E+05				
14	f [-]	87.0267084				
15						
16						

2. Define unknown variable: Here, we specify a cell to hold the unknown variable (discharge, Q) and provide an initial guess of $Q_0 = 1$.

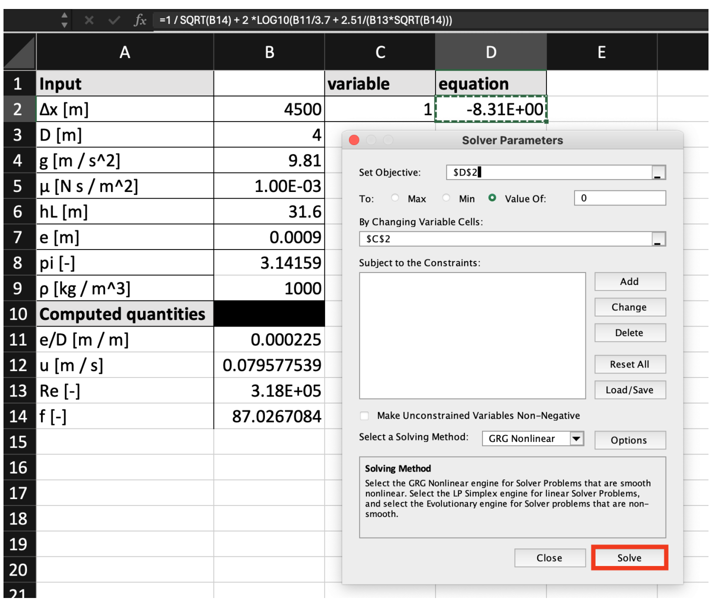
C2 | fx 1

	A	B	C	D	E	F
1	**Input**		**variable**	**equation**		
2	Δx [m]	4500	1	-8.31E+00		
3	D [m]	4				
4	g [m / s^2]	9.81				
5	μ [N s / m^2]	1.00E-03				
6	hL [m]	31.6				
7	e [m]	0.0009				
8	pi [-]	3.14159				
9	ρ [kg / m^3]	1000				
10	**Computed quantities**					
11	e/D [m / m]	0.000225				
12	u [m / s]	0.079577539				
13	Re [-]	3.18E+05				
14	f [-]	87.0267084				
15						

3. Define objective function: Here, we specify a cell to hold the objective function.



4a. Set up solver: Here, we click on the *Solver* icon under the *Data* tab in Excel. Note that the objective cell is `D2`, the variable cell is `C2`, and the solver is seeking to set the objective function to `Value of: 0`.



4b. Call solver: After clicking *Solve*, the solver populates the variable cell with the discharge needed to solve the objective function.

C2 fx 78.5491123232309

	A	B	C	D	E	F
1	**Input**		**variable**	**equation**		
2	Δx [m]	4500	78.54911232	8.31E-06		
3	D [m]	4				
4	g [m / s^2]	9.81				
5	μ [N s / m^2]	1.00E-03				
6	hL [m]	31.6				
7	e [m]	0.0009				
8	pi [-]	3.14159				
9	ρ [kg / m^3]	1000				
10	**Computed quantities**					
11	e/D [m / m]	0.000225				
12	u [m / s]	6.250745031				
13	Re [-]	2.50E+07				
14	f [-]	0.014104899				
15						
16						

The discharge in the pipe is thus:

$$\boxed{Q = 78.55 \ [m^3/s]} \tag{113}$$

Solving a system of nonlinear equations using Python

The above example can also be solved using a root-finding algorithm in a programming language like Python or MATLAB. The following code sample shows how the previous problem can be solved in Python.

```python
import numpy as np
from scipy.optimize import root_scalar
import matplotlib.pyplot as plt

D = 4           # Diameter [m]
rho = 1000      # Density of water [kg / m^3]
mu = 1e-3       # Dynamic viscosity [N s / m^2]
g = 9.81        # Acceleration due to gravity [m / s^2]
pi = np.pi      # Mathematical constant pi
e = 0.9 * 1e-3  # Roughness height [m]
dx = 4500       # Length of pipe [m]
ha = 8.3        # Pressure head at A [m]
hb = 76.7       # Pressure head at B [m]
dz = 100        # Elevation difference between A and B [m]

# Define function to find root of
def colebrook_white(Q):
   # Compute area of pipe
   A = pi * D**2 / 4
   # Compute velocity from flow rate
   u = Q / A
   # Compute Reynolds number
   Re = u * D * rho / mu
   # Compute expression for friction factor from energy equation
   f = (ha - hb + dz) * (2 * g * D) / (dx * u**2)
   # Minimize difference between LHS and RHS of Colebrook-White
   LHS = 1 / np.sqrt(f)
   RHS = -2 * np.log10(e / D / 3.7 + 2.51 / Re / np.sqrt(f))
   return LHS - RHS

# Find the root of the function
result = root_scalar(colebrook_white, method='bisect',
                     bracket=[1e-3, 100.])

# Print the discharge value that solves the equation
print(result.root)

>>> 78.54954371026334
```

Thus, the discharge that satisfies the set of equations is:

$$\boxed{Q = 78.55 \quad [m^3/s]} \tag{114}$$

The root-finding procedure may be visualized as follows:

```
# Plot Q vs. computed Q
plt.plot(Q, colebrook_white(Q), label='function')
plt.axhline(0, linestyle='--', color='k', label='x-axis')
plt.axvline(result.root, linestyle='--', color='r',
            label='solution')
plt.xlabel('Discharge, $Q$', size=14)
plt.ylabel(r'$ \frac{1}{\sqrt{f}} + 2 \cdot \log_{10} (
               \frac{e/D}{3.7} + \frac{2.51}{R_e \sqrt{f}} ) $',
           size=14)
plt.annotate(f'$Q =$ {result.root:.2f}', (result.root + 1, -1))
plt.legend()
plt.title('Numerically-determined root')
```

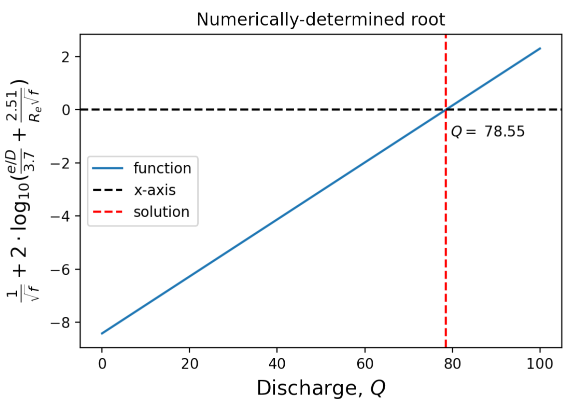


Figure 13: Solving the implicit Colebrook-White equation to determine discharge.

Empirical equations for steady pipe flow

While the Darcy-Weisbach equation for computing frictional losses is based on a well-established physical theory and is widely applicable, computing the friction factor via the Colebrook-White model is often challenging in practice. Thus, empirical equations derived from experimental data are often preferred for practical applications. These equations are often easier to use, but their applicability is limited.

Swamee-Jain approximation to Colebrook-White

The *Swamee-Jain approximation* provides a closed-form approximation of Colebrook-White equation for the friction factor f under turbulent flow conditions:

Swamee-Jain approximation to Colebrook-White equation

$$\boxed{f = \frac{0.25}{\log_{10}^2\left(\frac{e/D}{3.7} + \frac{5.74}{R_e^{0.9}}\right)}} \tag{115}$$

This approximation provides an accurate estimate ($\sim 1\%$) of the true Colebrook-White equation for:

- Typical roughness values: $10^{-6} < e/D < 10^{-2}$
- Typical Reynolds numbers: $5 \times 10^3 < R_e < 10^8$

Chezy-Manning equation

The Chezy-Manning equation is a popular alternative model for friction losses. While used mainly for open channel flow, it is also applicable to pressurized pipe flow. Recall from the *Chezy equation*, we have that:

$$u = C\sqrt{RS_f} \tag{116}$$

Where C is referred to as the *Chezy coefficient*. An empirical value for C was given by Robert Manning, based on a statistical average of different experimental studies:

$$C = \frac{\phi_{cm}}{n} R^{1/6} \tag{117}$$

Where n is the Manning roughness coefficient, and ϕ_{cm} is a constant that depends on the unit system employed. Substituting this formulation for C into the Chezy equation yields the so-called *Manning's equation*:

$$u = \frac{\phi_{cm}}{n} R^{2/3} S_f^{1/2} \tag{118}$$

More typically, Manning's equation is written in terms of the discharge by multiplying by the cross-sectional flow area.

Mannings equation (in terms of discharge), also referred to as the Chezy-Manning equation

$$\boxed{Q = \frac{\phi_{cm}}{n} A R^{2/3} S_f^{1/2}} \tag{119}$$

The constant ϕ changes depending on whether metric or U.S. customary units are used:

- For Q in cubic meters per second, A in square meters, R in meters (SI units):

$$\phi_{cm} = 1 \tag{120}$$

- For Q in cubic feet per second, A in square feet, R in feet (U.S. customary units):

$$\phi_{cm} = 1.49 \tag{121}$$

Limitations of Mannings equation

Mannings equation has a number of important limitations:

- It is only valid for water
- It is not a function of R_e
- It does not account for temperature or viscosity
- It is only valid for specific units (length in [m] for SI units, and [ft] for US customary units)

Hazen-Williams equation

The Hazen-Williams equation is another popular model for friction losses in pipes that is widely used for design of pipe systems in the US. Starting with the Chezy equation:

$$u = C\sqrt{RS_f} \tag{122}$$

If we allow the C coefficient to be $C = \phi_{hw} C_{hw} R^{0.13} S_f^{0.04}$, we arrive at the Hazen-Williams equation:

$$u = \phi_{hw} C_{hw} R^{0.63} S_f^{0.54} \tag{123}$$

Expressed in terms of discharge:

Hazen-Williams equation (in terms of discharge)

$$\boxed{Q = \phi_{hw} C_{hw} A R^{0.63} S_f^{0.54}} \tag{124}$$

The constant ϕ_{hw} changes depending on whether metric or U.S. customary units are used:

- For Q in cubic meters per second, A in square meters, R in meters (SI units):

$$\phi_{hw} = 0.849 \tag{125}$$

- For Q in cubic feet per second, A in square feet, R in feet (U.S. customary units):

$$\phi_{hw} = 1.318 \tag{126}$$

Limitations of Hazen-Williams equation

The Hazen-Williams equation is only valid for a specific range of pipe diameters and roughness conditions:

- Large pipe diameters: $D \geq 5$ $[cm]$
- Moderate velocities: $u \leq 3$ $[m/s]$

The Hazen-Williams equation has a number of important limitations:

- It is only valid for water
- It is not a function of R_e
- It does not account for temperature or viscosity
- It is only valid for specific units (length in [m] for SI units, and [ft] for US customary units)

Comparison of friction loss models

Each of the friction loss models may be written in the general form:

$$h_L = KQ^m \tag{127}$$

For Darcy-Weisbach, recall that:

$$h_L = f\frac{\Delta x}{D}\frac{u^2}{2g} \tag{128}$$

Thus, the Darcy-Weisbach equation may be rewritten:

$$h_L = K_{dw}Q^m \tag{129}$$

Where $m = 2$ and:

$$K_{dw} = f\frac{8\Delta x}{g\pi^2 D^5} \tag{130}$$

Similarly, for the Chezy-Manning equation:

$$Q = \frac{\phi_{cm}}{n} A R^{2/3} S_f^{1/2} \tag{131}$$

$$S_f^{1/2} = \frac{Qn}{\phi_{cm} A R^{2/3}} \tag{132}$$

$$S_f = \frac{Q^2 n^2}{\phi_{cm}^2 A^2 R^{4/3}} \tag{133}$$

Because $h_L = S_f \Delta x$, we can find the head loss by multiplying through by Δx:

$$h_L = \frac{Q^2 n^2 \Delta x}{\phi_{cm}^2 A^2 R^{4/3}} \tag{134}$$

Moreover, noting for a circular pipe that $A = \pi D^2/4$ and $R = A/P = D/4$, we have that Manning's equation can be expressed in the form:

$$h_L = K_{cm} Q^m \tag{135}$$

Where $m = 2$ and:

$$K_{cm} = \frac{4^{10/3} n^2 \Delta x}{\phi_{cm}^2 \pi^2 D^{16/3}} \tag{136}$$

For Hazen-Williams, a similar process can be carried out:

$$Q = \phi_{hw} C_{hw} A R^{0.63} S_f^{0.54} \tag{137}$$

$$S_f^{0.54} = \frac{Q}{\phi_{hw} C_{hw} R^{1.63}} \tag{138}$$

$$S_f = \frac{Q^{1.85}}{\phi_{hw}^{1.85} C_{hw}^{1.85} A^{1.85} R^{1.17}} \tag{139}$$

Thus, multiplying by Δx and assuming a circular pipe, we have:

$$h_L = K_{cm} Q^m \tag{140}$$

Where $m = 1.85$ and:

$$K_{cm} = \frac{4^{3.02} \Delta x}{\phi_{hw}^{1.85} C_{hw}^{1.85} \pi^{1.85} D^{4.87}} \tag{141}$$

Comparing the three friction loss models, we find that they can all be expressed in terms of the equation $h_L = KQ^m$, for varying values of K and m:

Equation	m	K (BG)	K (SI)
Darcy-Weisbach	2	$\frac{0.0252 f \Delta x}{D^5}$	$\frac{0.0826 f \Delta x}{D^5}$
Hazen-Williams	1.85	$\frac{4.73 \Delta x}{C_{hw}^{1.85} D^{4.87}}$	$\frac{10.7 \Delta x}{C_{hw}^{1.85} D^{4.87}}$
Manning	2	$\frac{4.64 n^2 \Delta x}{D^{5.33}}$	$\frac{10.3 n^2 \Delta x}{D^{5.33}}$

Table 5: Friction equations expressed in the form $h_L = KQ^m$.

Example 1.10

Problem: A horizontal pipe (cast iron) with a 10 [cm] diameter is 200 [m] long. Determine the discharge if the measured pressure head drop is 24.6 [m].

Solution: From our three friction loss equations, we have:

Darcy-Weisbach:

$$h_L = \frac{0.0826 f \Delta x}{D^5} Q^2 \quad \longrightarrow \quad Q_{dw} = \left(\frac{h_L D^5}{0.0826 f \Delta x} \right)^{1/2} \tag{142}$$

Hazen-Williams:

$$h_L = \frac{10.7 \Delta x Q^{1.85}}{C_{hw}^{1.85} D^{4.87}} \quad \longrightarrow \quad Q_{hw} = \left(\frac{h_L C_{hw}^{1.85} D^{4.87}}{10.7 \Delta x} \right)^{0.54} \tag{143}$$

Chezy-Manning:

$$h_L = \frac{10.3 n^2 \Delta x Q^2}{D^{5.33}} \quad \longrightarrow \quad Q_{cm} = \left(\frac{h_L D^{5.33}}{10.3 n^2 \Delta x} \right)^{1/2} \tag{144}$$

Solving each equation for Q, we get:

$$Q_{dw} = 0.024 \ [m^3/s] \tag{145}$$

$$0.013 \ [m^3/s] \leq Q_{hw} \leq 0.027 \ [m^3/s] \tag{146}$$

$$0.016 \ [m^3/s] \leq Q_{cm} \leq 0.022 \ [m^3/s] \tag{147}$$

It can be seen that the computed discharges are highly sensitive to the assumed roughness coefficients of the pipe material. In most cases, the uncertainty surrounding these roughness factors is much more significant than the uncertainty associated with choosing a particular empirical model.

Minor losses

Minor losses refer to energy losses in fluid flow resulting from expansions, contractions, or changes in direction. For our purposes, there are two main types:

- **Changes in geometry:** expansion, contraction, bends in pipes.
- **Hydraulic control structures:** check valves, gate valves, globe valves, etc.

The general equation for minor losses is:

$$h_\ell = K\frac{u^2}{2g} \tag{148}$$

Sudden contractions

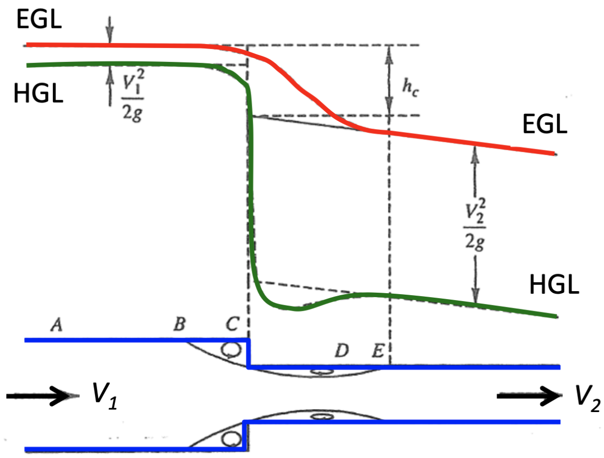
A sudden contraction in a pipe causes a drop in pressure due to both the increase in velocity as well as a loss of energy due to turbulence. The head loss due to a sudden contraction is given by the equation:

Head loss due to a sudden contraction

$$\boxed{h_{\ell c} = K_c\frac{u_2^2}{2g}} \tag{149}$$

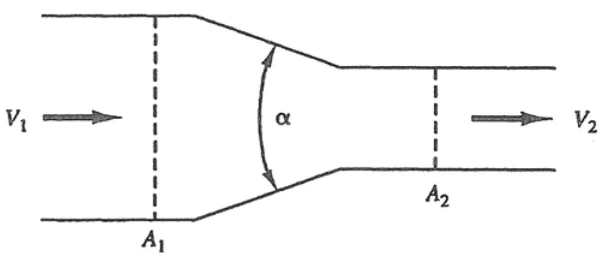
Where K_c is the contraction coefficient, and u_2 is the mean velocity in the smaller pipe after the contraction. The contraction coefficient K_c depends on the ratio of contraction D_2/D_1 and the velocity of flow.

The following diagram shows the head loss and pressure variation due to a sudden contraction. From points B to C, the HGL drops as the velocity increases and a region of stagnant water appears at the corner of contraction C. From point C to D, the streamlines separate from the pipe wall and form a high-speed jet that reattaches to the wall at point E. A majority of the energy loss occurs between points C and D in the *vena contracta*.



Gradual contractions

The head loss due to a pipe contraction can be greatly reduced through the introduction of a gradual pipe transition known as a confuser.



The head loss due to a confuser is given by the following equation:

Head loss due to a gradual contraction

$$\boxed{h_{\ell c'} = K_{c'} \frac{u_2^2}{2g}} \tag{150}$$

Where $K_{c'}$ is the confuser contraction coefficient, which varies with the transition angle α and the area ratio A_1/A_2.

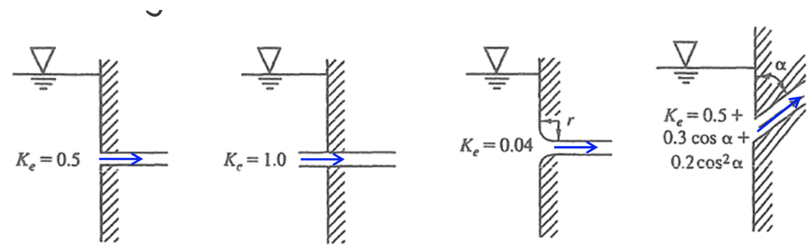
Entrance flow into pipe from large reservoir (contraction)

Another common type of contraction that causes head loss is the transition from a reservoir to pipe flow. The general formula for head loss at the entrance of a pipe is given as:

Head loss at pipe entrance

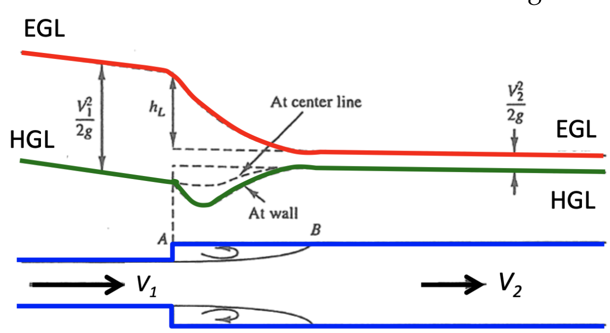
$$\boxed{h_{\ell e} = K_e \frac{u^2}{2g}} \tag{151}$$

Where u is the velocity in the pipe, and the entrance loss coefficient K_e depends on the entrance conditions, as shown below:



Sudden expansions

Pipe enlargements also cause a loss of energy due to the separation of streamlines and the formation of recirculating eddies.



The general expression for the loss in head due to a sudden expansion may be derived directly from momentum principles as:

Head loss due to sudden expansion

$$\boxed{h_{\ell E} = \frac{(u_1 - u_2)^2}{2g}} \tag{152}$$

Where u_1 is the velocity upstream of the expansion and u_2 is the velocity downstream of the expansion. A brief derivation follows here. First, applying an energy balance between the upstream and downstream ends of the expansion, we have:

$$\frac{p_1}{\gamma} + \frac{u_1^2}{2g} = \frac{p_2}{\gamma} + \frac{u_2^2}{2g} + h_\ell \tag{153}$$

$$h_\ell = \frac{p_1 - p_2}{\gamma} + \frac{u_1^2 - u_2^2}{2g} \tag{154}$$

Applying a momentum balance between the upstream and downstream ends of the contraction, we have that the change in momentum between the downstream and upstream ends is equal to the pressure force exerted by the upstream fluid plus the reaction force exerted by the wall minus the pressure force exerted by the downstream fluid:

$$\rho Q(u_2 - u_1) = p_1 A_1 - p_2 A_2 + p_1(A_2 - A_1) \tag{155}$$

From continuity, we have that $Q = u_1 A_1 = u_2 A_2$:

$$\rho(A_2 u_2^2 - A_1 u_1^2) = p_1 A_1 - p_2 A_2 + p_1(A_2 - A_1) \tag{156}$$

Expanding the right-hand side, we have:

$$\rho(A_2 u_2^2 - A_1 u_1^2) = p_1 A_1 - p_2 A_2 + p_1 A_2 - p_1 A_1 \tag{157}$$

$$\rho(A_2 u_2^2 - A_1 u_1^2) = A_2(p_1 - p_2) \tag{158}$$

$$u_2^2 - \frac{A_1}{A_2} u_1^2 = \frac{p_1 - p_2}{\rho} \tag{159}$$

From continuity:

$$u_1 A_1 = u_2 A_2 \tag{160}$$

$$u_2 = \frac{A_1}{A_2} u_1 \tag{161}$$

Thus:

$$u_2^2 - u_2 u_1 = \frac{p_1 - p_2}{\rho} \tag{162}$$

$$u_2(u_2 - u_1) = \frac{p_1 - p_2}{\rho} \tag{163}$$

$$\frac{u_2(u_2 - u_1)}{g} = \frac{p_1 - p_2}{\gamma} \tag{164}$$

Substituting this expression back into the equation for the head loss, we have:

$$h_\ell = \frac{u_2(u_2 - u_1)}{g} + \frac{u_1^2 - u_2^2}{2g} \tag{165}$$

$$h_\ell = \frac{2u_2(u_2 - u_1) + u_1^2 - u_2^2}{2g} \tag{166}$$

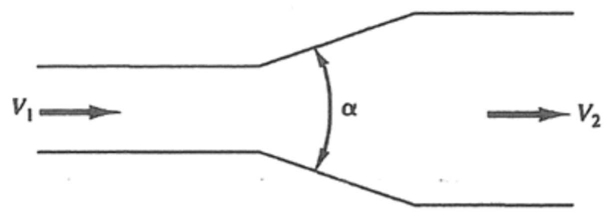
$$h_\ell = \frac{2u_2^2 - 2u_1 u_2 + u_1^2 - u_2^2}{2g} \tag{167}$$

$$h_\ell = \frac{u_1^2 - 2u_1 u_2 + u_2^2}{2g} \tag{168}$$

$$h_\ell = \frac{(u_1 - u_2)^2}{2g} \tag{169}$$

Gradual expansions

The head losses due to a sudden expansion may be drastically reduced through the introduction of a gradual pipe transition known as a *diffuser*.



The general expression for head loss in a diffuser is given by:

Head loss due to gradual expansion

$$h_{\ell E'} = K_{E'} \frac{(u_1 - u_2)^2}{2g} \tag{170}$$

Where the diffuser expansions coefficient $K_{E'}$ depends on the transition angle α.

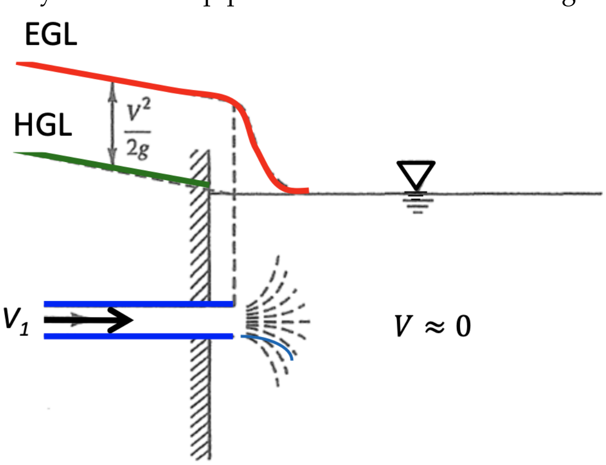
Exit flow into large reservoir from pipe (expansion)

A submerged pipe discharging into a large reservoir is a special case of head loss due to enlargement, in which the velocity at the downstream end of the expansion can be taken to be zero:

Head loss at pipe exit

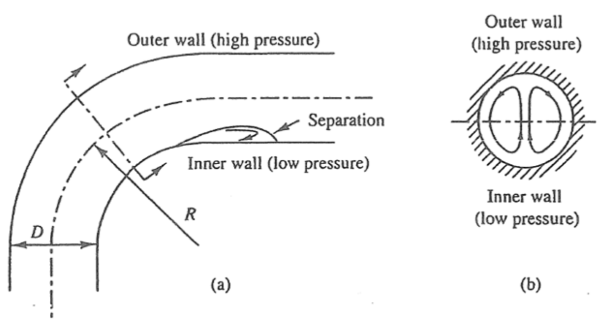
$$h_{\ell d} = \frac{u^2}{2g} \tag{171}$$

Where u is the velocity of the pipe. In other words, the entire velocity head of the pipe flow is lost to the discharge head loss.



Head loss in pipe bends

Pipe bends cause a separation of streamlines from the inner wall of the bend. This separation of streamlines causes turbulent eddies that dissipate energy and may persist as far as 100 pipe diameters downstream of the bend.



The general expression for the head loss around a pipe bend is given by:

Head loss around pipe bend

$$h_{\ell b} = K_b \frac{u_2^2}{2g} \tag{172}$$

Where the bend loss coefficient K_b depends on the ratio of the bend radius to pipe diameter R/D.

Head loss in valves

Valves are installed in water distribution networks to control pressure and flow rates. Depending on how the valve is designed, head losses may occur even if the valve is completely open. The head loss within a valve is expressed as:

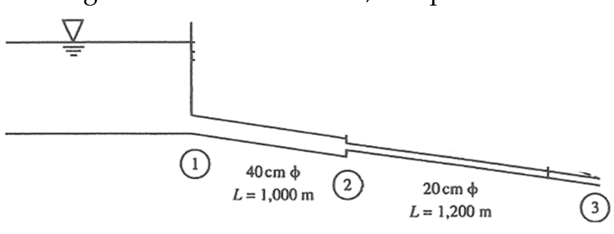
Head loss across valve

$$h_{\ell v} = K_v \frac{u^2}{2g} \tag{173}$$

Where K_v is the valve loss coefficient, and u is the velocity of flow through the valve.

EXAMPLE 1.11

Problem: Cast iron pipe transports water from a reservoir and discharges into the air through a rotary valve at $\Delta z = 100$ $[m]$ below the water surface elevation. If the water temperature is 10 °C and square-edge connections are used, compute the flow rate in the pipe.



Solution: Applying the energy equation between the top of the reservoir (0) and the outlet (3), we have:

$$\frac{p_0}{\gamma} + z_0 + \frac{u_0^2}{2g} = \frac{p_3}{\gamma} + z_3 + \frac{u_3^2}{2g} + h_L \tag{174}$$

$$\Delta z = \frac{u_3^2}{2g} + h_L \tag{175}$$

Assuming that the valve is completely open, and the same diameter as pipe 2, we have that $u_2 = u_3$:

$$\Delta z = \frac{u_2^2}{2g} + h_L \tag{176}$$

The total head losses h_L are equal to the frictional losses plus the minor losses:

$$h_L = h_{major} + h_{minor} \tag{177}$$

Using Darcy-Weisbach, the major losses are equal to the frictional losses in pipe 1 plus the frictional losses in pipe 2.

$$h_{major} = f_1 \frac{\Delta x_1}{D_1} \frac{u_1^2}{2g} + f_2 \frac{\Delta x_2}{D_2} \frac{u_2^2}{2g} \tag{178}$$

The minor losses in turn comprise the entrance, contraction, and valve losses.

$$h_{minor} = h_{entrance} + h_{contraction} + h_{valve} \tag{179}$$

$$h_{minor} = K_{\ell e} \frac{u_1^2}{2g} + K_{\ell c} \frac{u_2^2}{2g} + K_{\ell v} \frac{u_2^2}{2g} \tag{180}$$

Thus, we have that:

$$\Delta z = \frac{u_2^2}{2g} + f_1 \frac{\Delta x_1}{D_1} \frac{u_1^2}{2g} + f_2 \frac{\Delta x_2}{D_2} \frac{u_2^2}{2g} + K_{\ell e} \frac{u_1^2}{2g} + K_{\ell c} \frac{u_2^2}{2g} + K_{\ell v} \frac{u_2^2}{2g} \tag{181}$$

Simplifying, we have:

$$\Delta z = \left(f_1 \frac{\Delta x_1}{D_1} + K_{\ell e} \right) \frac{u_1^2}{2g} + \left(1 + f_2 \frac{\Delta x_2}{D_2} + K_{\ell c} + K_{\ell v} \right) \frac{u_2^2}{2g} \tag{182}$$

From continuity, we have $Q = A_1 u_1 = A_2 u_2$. Thus:

$$\Delta z = \left(f_1 \frac{\Delta x_1}{D_1} + K_{\ell e} \right) \frac{Q^2}{2gA_1^2} + \left(1 + f_2 \frac{\Delta x_2}{D_2} + K_{\ell c} + K_{\ell v} \right) \frac{Q^2}{2gA_2^2} \tag{183}$$

$$0 = \frac{Q^2}{2g} \left[\left(f_1 \frac{\Delta x_1}{D_1} + K_{\ell e} \right) \frac{1}{A_1^2} + \left(1 + f_2 \frac{\Delta x_2}{D_2} + K_{\ell c} + K_{\ell v} \right) \frac{1}{A_2^2} \right] - \Delta z \tag{184}$$

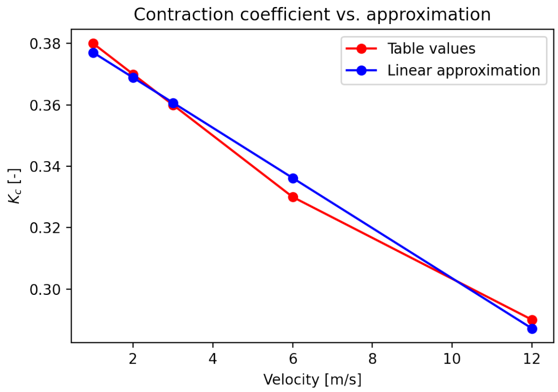
We know that for an entrance without a transition $K_{\ell e} = 0.5$ and for a rotary valve, $K_v = 10$. The contraction coefficient depends on the velocity in the pipe, which we do not know, but we will assume for the first trial that the velocity in the smaller pipe is around 6 $[m/s]$ and thus the loss coefficient is around $K_{\ell c} = 0.33$. We can thus use an iterative procedure to solve the problem, as before.

Importing our modules:

```
import numpy as np
from scipy.optimize import root_scalar
import matplotlib.pyplot as plt

D_1 = 0.4      # Diameter of pipe 1 [m]
D_2 = 0.2      # Diameter of pipe 2 [m]
dx_1 = 1000    # Length of pipe 1 [m]
dx_2 = 1200    # Length of pipe 2 [m]
Ke = 0.5       # Entrance loss coefficient
Kv = 10        # Valve loss coefficient [-]
e = 0.26 * 1e-3 # Roughness height [m]
rho = 1000     # Density of water [kg / m^3]
mu = 1e-3      # Dynamic viscosity [N s / m^2]
g = 9.81       # Acceleration due to gravity [m / s^2]
pi = np.pi     # Mathematical constant pi
dz = 100       # Elevation difference between A and B [m]
```

First, we will create a function to compute the contraction coefficient based on the velocity. To accomplish this, let's fit a polynomial to the values given in the table:

```
# Fit linear function to contraction coefficients
velocities = [1, 2, 3, 6, 12]
contraction_coeffs = [0.38, 0.37, 0.36, 0.33, 0.29]
m, b = np.polyfit(velocities, contraction_coeffs, 1)

# Plot the result
plt.plot(velocities, contraction_coeffs,
       label='Table values', c='r', marker='o')
plt.plot(velocities, m * np.array(velocities) + b,
       label='Linear approximation', c='b', marker='o')
plt.legend()
plt.xlabel('Velocity [m/s]')
plt.ylabel('$K_c$ [-]')
plt.title('Contraction coefficient vs. approximation')
```



Next, we can define functions to compute the friction factor and the contraction coefficient:

```
# Define a function to compute the friction factor
def swamee_jain(e, D, Re):
    f = 0.25 / np.log10(e / D / 3.7 + 5.74 / Re**0.9)**2
    return f

def Kc_function(u):
    return m * u + b
```

Finally, we can solve for the discharge using the bisection method:

```
# Define function to find root of
def solve(Q):
    # Compute area of pipe
    A_1 = pi * D_1**2 / 4
    A_2 = pi * D_2**2 / 4
    # Compute velocity from flow rate
    u_1 = Q / A_1
    u_2 = Q / A_2
    # Compute Reynolds number
    Re_1 = u_1 * D_1 * rho / mu
    Re_2 = u_1 * D_1 * rho / mu
    # Compute expression for friction factor
    f_1 = swamee_jain(e, D_1, Re_1)
    f_2 = swamee_jain(e, D_2, Re_2)
    # Compute contraction coefficient
    Kc = Kc_function(u_2)
    # Minimize difference between LHS and RHS of Colebrook-White
    result = (Q**2 / 2 / g) * ((f_1 * dx_1 / D_1 + Ke) / A_1**2
                    + (f_2 * dx_2 / D_2 + Kc + Kv) / A_2**2) -
                        dz
    return result

# Find the root of the function
result = root_scalar(solve, method='bisect',
                bracket=[1e-5, 100.])
```

```python
# Print the discharge value that solves the equation
print(result.root)

>>> 0.11601685139078272
```

Thus, the discharge is approximately:

$$\boxed{Q = 0.116 \quad [m^3/s]} \tag{185}$$

Extra topics

Negative pressures

Negative pressures can develop when pipelines are raised above the hydraulic grade line. Negative pressures can cause a number of problems in pipeline systems, such as:

- *Cavitation:* When water in a pressurized pipe drops below the vapor pressure of water, water will spontaneously vaporize to form vapor pockets (cavitation) that separate the water in the pipe. When these vapor pockets collapse in regions of higher pressure downstream, they cause vibrations and sound that can greatly damage the walls of the pipe.
- *Contaminant intrusion:* Negative pressures within a pipe can cause pipes to draw in contaminants from water in the soil surrounding the pipe.

To avoid cavitation, engineers will typically design pipe systems such that negative pressures are at least 2/3 of the vapor pressure of water. The vapor pressure of water is:

$$P_{abs,v} = 2.37 \times 10^3 \quad [Pa] \tag{186}$$

Likewise, atmospheric pressure is:

$$P_{atm} = 1.014 \times 10^5 \quad [Pa] \tag{187}$$

Thus, the negative pressure at which cavitation will occur is:

$$P_{gage} = P_{abs,v} - P_{atm} \tag{188}$$

$$= 2.37 \times 10^3 - 1.014 \times 10^5 = -99 \times 10^3 \quad [Pa] \tag{189}$$

In units of head:

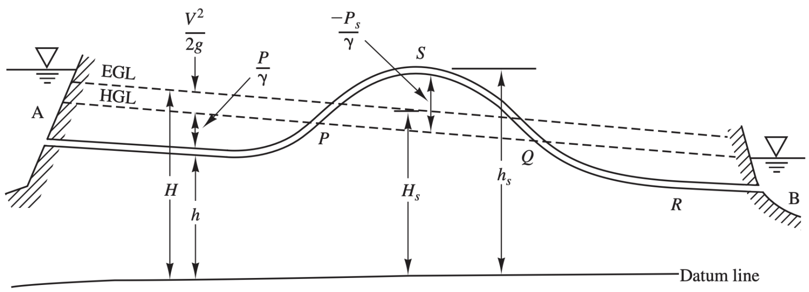
$$\frac{p_{gage}}{\gamma} = -10.2 \quad [m] \tag{190}$$

Thus, to avoid cavitation we want to ensure a minimum pressure of:

$$\frac{p_{cav}}{\gamma} = \frac{2}{3}\frac{p_{gage}}{\gamma} \approx -7 \quad [m] \tag{191}$$

EXAMPLE 1.12

Problem: A uniform steel pipeline 40 [cm] in diameter and 2000 [m] long carries water at 10 °C between two reservoirs. The two reservoirs have a water surface elevation difference of 30 [m]. At midlength the pipeline must be raised to carry the water over a small hill. Determine the maximum elevation to which the summit S may be raised above the lower reservoir water surface.



Solution: The discharge and velocity of water may be determined using the discharge equation:

$$h_A - h_B = 30\ [m] = \left(K_e + f\frac{\Delta x}{D} + K_d\right)\frac{u^2}{2g} \tag{192}$$

Assuming a square entrance and solving iteratively for the friction factor f and velocity u:

$$30\ [m] = \left(0.5 + f\frac{2000}{0.4} + 1\right)\frac{u^2}{2(9.81)} \tag{193}$$

$$f = 0.014 \tag{194}$$

$$u = 2.87\ [m/s] \tag{195}$$

At 10 $°C$, the water vapor pressure head is -10.2 [m]. Applying the energy equation between the summit S and the lower reservoir we have:

$$\frac{u_s^2}{2g} + \frac{p_s}{\gamma} + h_{rs} = \frac{u_d^2}{2g} + \frac{p_d}{\gamma} + h_r + \left(f\frac{\Delta x}{D} + K_d\right)\frac{u_s^2}{2g} \qquad (196)$$

$$\frac{u_s^2}{2g} + \frac{p_s}{\gamma} + h_{rs} = h_r + \left(f\frac{\Delta x}{D} + K_d\right)\frac{u_s^2}{2g} \qquad (197)$$

$$h_{rs} - h_r = -\frac{p_s}{\gamma} + \left(f\frac{\Delta x}{D} + K_d - 1\right)\frac{u_s^2}{2g} \qquad (198)$$

$$h_{rs} - h_r = 10.2 + \left(0.014\frac{1000}{0.4} + 1 - 1\right)\frac{2.87^2}{2(9.81)} \qquad (199)$$

$$\boxed{h_{rs} - h_r = 24.9 \quad [m]} \qquad (200)$$

Part III

Networked pipe flow

Flow in pipe networks

General pipe network problem

The general problem for pipe networks is as follows. Given the following parameters:

- Physical data (pipe diameter, length, roughness)
- Network layout
- End-user demands
- Operating conditions (water levels in reservoirs)

We are tasked with determining the internal states of the water distribution network, including:

- Flows and velocities in pipes
- Pressures at nodes (end-users)

General solution procedure for pipe networks

To solve for internal flows and heads in pipe networks, the basic idea is to apply the continuity equation around each junction and the energy equation across each pipe.

Continuity

First, the continuity equation is applied around each junction. Conceptually, this means that the sum of pipe flows going into each junction is equal to the sum of the pipe flows going out plus the external demand flow from the junction.

$$\begin{aligned}\text{[Pipe flow into junction]} = &\ \text{[Pipe flow out of junction]} \\ &+\text{[External demand from junction]}\end{aligned} \tag{201}$$

Note that flows can be defined to be either positive or negative depending on their direction of flow. Thus, we can write the continuity equation mathematically for a junction i connected to multiple pipes $k \in S$ as follows:

Continuity as applied to a junction in a pipe network

$$\boxed{0 = \sum_{k \in S_i} Q_k - Q_{dem,i}} \tag{202}$$

Where S_i is the set of pipes connected to junction i, Q_k is the flow in pipe k connected to junction i, and $Q_{dem,i}$ is the external demand from junction i.

Energy

Next, the energy equation is applied across each pipe. Stated conceptually:

$$\begin{aligned}\text{[Head at upstream junction]} &= \text{[Head at downstream junction]} \\ &+ \text{[Head loss through connecting pipe]}\end{aligned} \tag{203}$$

For pipe k connecting upstream junction i with downstream junction j, we can write this expression mathematically as:

$$H_i = H_j + h_{L,k} \tag{204}$$

Where H_i is the total head at the upstream junction, H_j is the total head at the downstream junction, and $h_{L,k}$ is the head loss across the pipe. Rearranging to set the left-hand side equal to zero:

Energy equation as applied to a pipe in a pipe network

$$\boxed{0 = H_i - H_j - h_{L,k}} \tag{205}$$

Solving equations simultaneously

To determine the internal heads H_i and internal flows Q_k, we need to simultaneously satisfy the energy balance for all pipes and the continuity equation for all junctions. One way to accomplish this task is to minimize the sum of squares of the right-hand sides of our continuity and energy equations. First define a slack variable ϵ that represents the value of the right-hand side of the equation under the current guesses for flow rates and heads:

$$\epsilon_{c,i} = \sum_{k \in S_i} Q_k - Q_{dem,i} \tag{206}$$

$$\epsilon_{e,k} = H_i - H_j - h_{L,k} \tag{207}$$

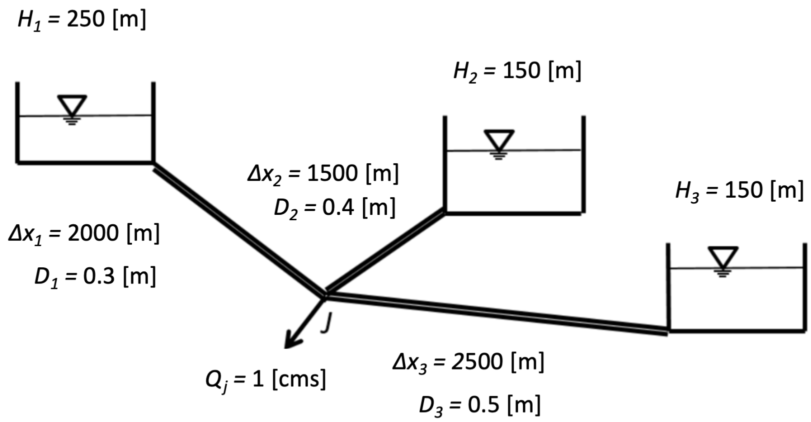
We can then select values of Q and H that minimize the sum of squares of these residuals.

Quadratic objective function for solving pipe network problems

$$\boxed{\min_{H_i, Q_k} \sum_{i=1}^{M} \epsilon_{c,i}^2 + \sum_{j=1}^{N} \epsilon_{e,j}^2} \tag{208}$$

Branched pipe networks

Consider the following branched pipe network consisting of three reservoirs connected to a central junction. At the central junction j, there is an external demand of $Q_j = 1$ $[m^3/s]$. For all three pipes, the roughness is described by Hazen-Williams coefficient $C_{HW} = 100$. Pipe parameters are described in the diagram below:



Writing out the continuity equation junction j:

$$\epsilon_{cj} = Q_1 + Q_2 + Q_3 - Q_j \tag{209}$$

Where $\epsilon_{cj} = 0$ when continuity is satisfied. Writing out the energy equation for each link:

$$\epsilon_{e1} = H_1 - H_j - K_1 Q_1 |Q_1|^{0.85} \tag{210}$$
$$\epsilon_{e2} = H_2 - H_j - K_2 Q_2 |Q_2|^{0.85} \tag{211}$$
$$\epsilon_{e3} = H_3 - H_j - K_3 Q_3 |Q_3|^{0.85} \tag{212}$$

Where $\epsilon_{ei} = 0$ when the energy balance is satisfied, and where K_i is defined according to the Hazen-Williams formulation:

$$K_i = \frac{10.7 \Delta x_i}{D_i^{4.87} C_{HW}^{1.85}} \tag{213}$$

For our objective function, we want to select Q_1, Q_2, Q_3, H_j to minimize the residuals $\epsilon_{cj}, \epsilon_{e1}, \epsilon_{e2}, \epsilon_{e3}$

$$\min_{Q_1, Q_2, Q_3, H_j} \epsilon_{cj}^2 + \epsilon_{e1}^2 + \epsilon_{e2}^2 + \epsilon_{e3}^2 \tag{214}$$

Using Excel, we can set the solver to minimize this sum of residuals.[1] The steps to follow in Excel are given as follows:

[1] Note that under solver options, you will want to change derivatives from 'forward' to 'central')

1. *Specify parameters:* Fill in parameter values.

D2

	A	B	C	D	E
1	**Input**		**Variables**		**Objective**
2	H_R1	250	H_j		
3	H_R2	150	Q_1		
4	H_R3	150	Q_2		
5	dx_1	2000	Q_3		
6	dx_2	1500	**Equations**		
7	dx_3	2500	Energy 1		
8	D_1	0.3	Energy 2		
9	D_2	0.4	Energy 3		
10	D_3	0.5	Continuity j		
11	Q_J	1			
12	C_hw	100			

2. *Specify initial guesses:* Fill in initial guesses for all unknown heads and flows.

D7

	A	B	C	D	E
1	**Input**		**Variables**		**Objective**
2	H_R1	250	H_j	100	
3	H_R2	150	Q_1	0.3	
4	H_R3	150	Q_2	0.3	
5	dx_1	2000	Q_3	0.3	
6	dx_2	1500	**Equations**		
7	dx_3	2500	Energy 1		
8	D_1	0.3	Energy 2		
9	D_2	0.4	Energy 3		
10	D_3	0.5	Continuity j		
11	Q_J	1			
12	C_hw	100			

3. *Specify energy equation for each pipe:*

D9 =B4-D2 - SIGN(D5) * 10.7 * B7 / B12^1.85 / B10^4.87 * ABS(D5)^1.85

	A	B	C	D	E
1	**Input**		**Variables**		**Objective**
2	H_R1	250	H_j	100	
3	H_R2	150	Q_1	0.3	
4	H_R3	150	Q_2	0.3	
5	dx_1	2000	Q_3	0.3	
6	dx_2	1500	**Equations**		
7	dx_3	2500	Energy 1	-11.997228	
8	D_1	0.3	Energy 2	20.0692775	
9	D_2	0.4	Energy 3	33.1727097	
10	D_3	0.5	Continuity j		
11	Q_J	1			
12	C_hw	100			

The energy equations for each link are specified as follows:

```
=B2-$D$2 - SIGN(D3) * 10.7 * B5 / $B$12^1.85 / B8^4.87 *
ABS(D3)^1.85
=B3-$D$2 - SIGN(D4) * 10.7 * B6 / $B$12^1.85 / B9^4.87 *
ABS(D4)^1.85
=B4-$D$2 - SIGN(D5) * 10.7 * B7 / $B$12^1.85 / B10^4.87 *
ABS(D5)^1.85
```

4. *Specify continuity equation for each junction:*

D10 =B11-D3-D4-D5

	A	B	C	D	E
1	**Input**		**Variables**		**Objective**
2	H_R1	250	H_j	100	
3	H_R2	150	Q_1	0.3	
4	H_R3	150	Q_2	0.3	
5	dx_1	2000	Q_3	0.3	
6	dx_2	1500	**Equations**		
7	dx_3	2500	Energy 1	-11.997228	
8	D_1	0.3	Energy 2	20.0692775	
9	D_2	0.4	Energy 3	33.1727097	
10	D_3	0.5	Continuity j	0.1	
11	Q_J	1			
12	C_hw	100			

The continuity equation for junction j is specified as follows:

```
=B11-D3-D4-D5
```

5. *Specify objective function:*

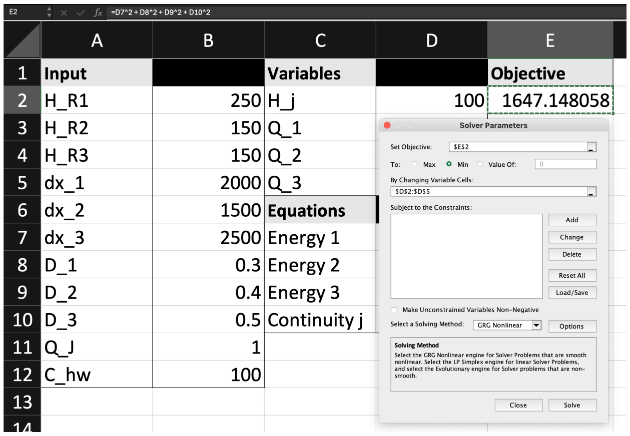
E2 =D7^2 + D8^2 + D9^2 + D10^2

	A	B	C	D	E
1	**Input**		**Variables**		**Objective**
2	H_R1	250	H_j	100	1647.148058
3	H_R2	150	Q_1	0.3	
4	H_R3	150	Q_2	0.3	
5	dx_1	2000	Q_3	0.3	
6	dx_2	1500	**Equations**		
7	dx_3	2500	Energy 1	-11.997228	
8	D_1	0.3	Energy 2	20.0692775	
9	D_2	0.4	Energy 3	33.1727097	
10	D_3	0.5	Continuity j	0.1	
11	Q_J	1			
12	C_hw	100			

For our objective function, we can use:

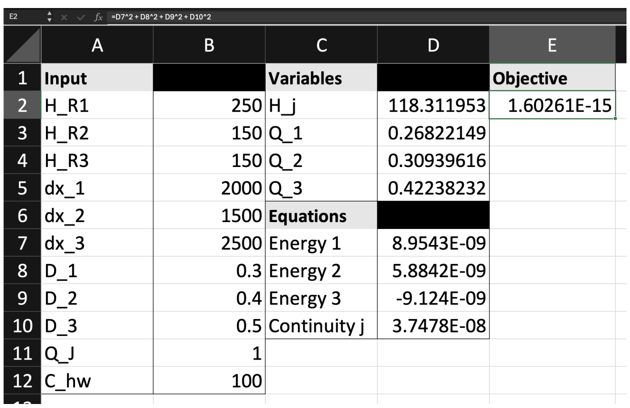
```
=D7^2 + D8^2 + D9^2 + D10^2
```

6. Invoke Excel Solver: Set the Excel Solver to minimize our objective function.


E2 =D7^2+D8^2+D9^2+D10^2

	A	B	C	D	E
1	**Input**		**Variables**		**Objective**
2	H_R1	250	H_j	100	1647.148058
3	H_R2	150	Q_1		
4	H_R3	150	Q_2		
5	dx_1	2000	Q_3		
6	dx_2	1500	**Equations**		
7	dx_3	2500	Energy 1		
8	D_1	0.3	Energy 2		
9	D_2	0.4	Energy 3		
10	D_3	0.5	Continuity j		
11	Q_J	1			
12	C_hw	100			
13					

6. Run solver until convergence:


E2 =D7^2+D8^2+D9^2+D10^2

	A	B	C	D	E
1	**Input**		**Variables**		**Objective**
2	H_R1	250	H_j	118.311953	1.60261E-15
3	H_R2	150	Q_1	0.26822149	
4	H_R3	150	Q_2	0.30939616	
5	dx_1	2000	Q_3	0.42238232	
6	dx_2	1500	**Equations**		
7	dx_3	2500	Energy 1	8.9543E-09	
8	D_1	0.3	Energy 2	5.8842E-09	
9	D_2	0.4	Energy 3	-9.124E-09	
10	D_3	0.5	Continuity j	3.7478E-08	
11	Q_J	1			
12	C_hw	100			

Applying the solver in excel, we find that the following values satisfy our energy and continuity balances:

$$Q_1 = 0.268\ [m^3/s] \qquad Q_2 = 0.309\ [m^3/s]$$
$$Q_3 = 0.422\ [m^3/s] \qquad H_j = 118.32\ [m]$$

The solution can also be obtained using a numeric solver in a programming language like Python:

```
import numpy as np
from scipy.optimize import root
import matplotlib.pyplot as plt

D_1 = 0.3      # Diameter of pipe 1 [m]
D_2 = 0.4      # Diameter of pipe 2 [m]
D_3 = 0.5      # Diameter of pipe 3 [m]
dx_1 = 2000    # Length of pipe 1 [m]
dx_2 = 1500    # Length of pipe 2 [m]
dx_3 = 2500    # Length of pipe 3 [m]
H_1 = 250      # Head at junction 1 [m]
H_2 = 150      # Head at junction 2 [m]
H_3 = 150      # Head at junction 3 [m]
C_hw = 100     # Hazen-Williams coefficient
K = 10.7       # Constant for Hazen-Williams equation
Q_j = 1.       # Outflow [m^3 / s]

# Define Hazen-Williams function
def hazen_williams(Q, D, dx, C_hw, K=10.7):
   h_L = (np.sign(Q) * K * dx / C_hw**1.85 / D**4.87 *
        np.abs(Q)**1.85)
   return h_L

# Define objective function
def objective_function(x):
   Q_1, Q_2, Q_3, H_j = x
   h_L1 = hazen_williams(Q_1, D_1, dx_1, C_hw, K)
   h_L2 = hazen_williams(Q_2, D_2, dx_2, C_hw, K)
   h_L3 = hazen_williams(Q_3, D_3, dx_3, C_hw, K)
   eq_1 = H_1 - H_j - h_L1
   eq_2 = H_2 - H_j - h_L2
   eq_3 = H_3 - H_j - h_L3
   eq_4 = Q_1 + Q_2 + Q_3 - Q_j
   result = [eq_1, eq_2, eq_3, eq_4]
   return result

# Initial guess
x0 = [0.3, 0.3, 0.4, 10.]

# Call solver
solution = root(objective_function, x0)
```

Invoking the solution, we get:

```
>>> solution

 message: The solution converged.
 success: True
  status: 1
     fun: [ 3.695e-13 -9.308e-13 1.222e-12 0.000e+00]
       x: [ 2.682e-01 3.094e-01 4.224e-01 1.183e+02]
  method: hybr
    nfev: 11
    fjac: [[-9.999e-01 1.041e-02 -6.914e-03 1.094e-03]
           [-1.053e-02 -9.997e-01 1.932e-02 5.342e-03]
           [ 6.720e-03 -1.935e-02 -9.998e-01 7.588e-03]
           [-1.099e-03 -5.477e-03 -7.491e-03 -1.000e+00]]
       r: [ 9.144e+02 -1.395e+00 2.134e+00 9.833e-01 1.875e+02
           -3.888e+00 9.760e-01 1.342e+02 1.030e+00 1.410e-02]
     qtf: [ 1.973e-08 -5.305e-08 6.666e-08 2.440e-10]
```

Part IV

Pumps

Overview of pumps

Pumps convert *mechanical energy* into *hydraulic energy*. They are used to move water from a lower elevation to a higher elevation, or to keep water distribution systems pressurized. Within water supply and distribution systems, pumps are used for a number of specific applications [2]:

- Pumping into an elevated storage tank from a source such as a well
- Pumping raw water from a river or lake
- Pumping (high-service pumping) of finished water at high pressure
- Booster pumping
  - In-line booster pumping into an elevated tank or reservoir
  - Distribution system booster pumping without a storage tank

The action of pumps on a hydraulic system is perhaps best illustrated by visualizing the energy and hydraulic grade lines.

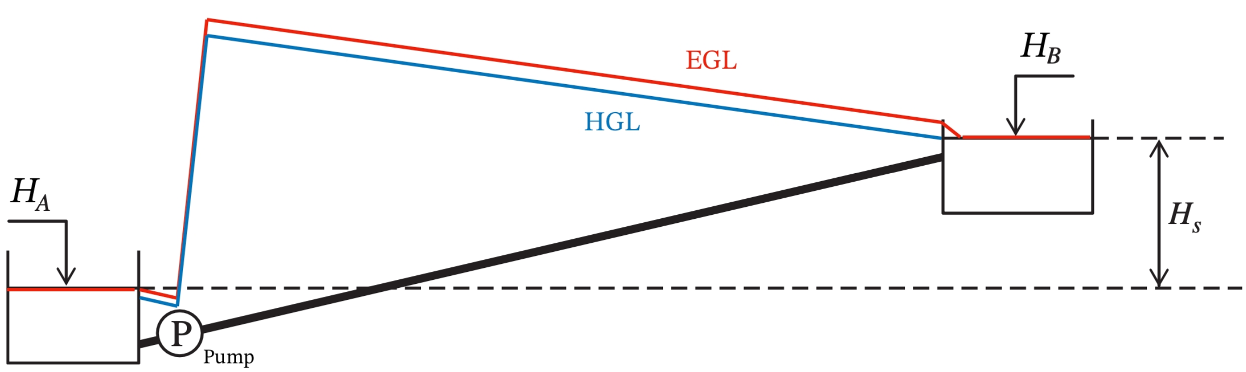


Figure 14: Action of pump on hydraulic grade line and energy grade line.

Types of pumps

Pumps can be divided into three basic types:

- **Direct lift (i.e. turbo-hydraulic pumps):** These pumps include centrifugal, propeller, and jet pumps. They move fluids with a rotating vane or another moving fluid that imparts energy to the fluid.
- **Positive displacement:** These pumps include screw, piston, and gear pumps. They work by trapping the fluid into a compartment and then displacing a fixed amount of it into the discharge pipe using a rotary or reciprocating mechanism.
- **Gravity:** These pumps include hydraulic ram pumps. They function like a hydraulic transformer, taking water at a low hydraulic head and high flow rate and discharging water at higher hydraulic head and lower flow rate by taking advantage of the water hammer effect. They require no outside source of power.

Energy increase through pumps

In this class, we will focus on direct-lift pumps. Direct-lift pumps add energy to the fluid system through the work performed by the pump impeller on the fluid. Applying an energy balance around the upstream and downstream ends of the pump, we have that the head at the downstream end (2) is equal to the head at the upstream end (1) plus the head added by the pump:

$$H_1 + H_p = H_2 \tag{215}$$

Expanding the terms for the total head upstream and downstream of the pump, we have that:

$$\frac{p_1}{\gamma} + z_1 + \frac{u_1^2}{2g} + H_p = \frac{p_2}{\gamma} + z_2 + \frac{u_2^2}{2g} \tag{216}$$

Thus, the energy added by the pump is equal to:

Pump head (i.e. energy head added by pump)

$$\boxed{H_p = \frac{p_2 - p_1}{\gamma} + (z_2 - z_1) + \frac{u_2^2 - u_1^2}{2g}} \tag{217}$$

In practical terms, however, a majority of the added head will be in terms of the pressure head, and thus the head added by the pump will be roughly equal to the pressure head difference between the upstream and downstream ends:

$$H_p \approx \frac{p_2 - p_1}{\gamma} \tag{218}$$

Pump operating points

Pump design basics

Returning to the two-reservoir problem. If we apply the energy equation between points A and B, and neglect minor losses, we have:

$$H_A + H_p = H_B + h_L \tag{219}$$

Where H_A is the total head in reservoir A, H_B is the total head in reservoir B, H_p is the head provided by the pump, and h_L is the head loss due to friction between reservoirs A and B. This means that the head provided by the pump is given by:

$$H_p = (H_B - H_A) + h_L \tag{220}$$

$$= H_{stat} + h_L \tag{221}$$

Where the required elevation rise H_{stat} is referred to as the *static head*. Note that the static head $H_{stat} = (H_B - H_A)$ is a constant, whereas the head loss h_L depends on the flow rate Q. If more flow is pushed through the system, then more friction losses are incurred and a greater pump head is required. The quantity $H_{sys} = H_{stat} + h_L$ is referred to as the *system head* requirement.

System head (i.e. head that pump must satisfy for a given flow rate)

$$\boxed{H_{sys} = H_{stat} + h_L} \tag{222}$$

Using any friction loss equation (e.g. Darcy-Weisbach, Hazen-Williams, Manning) will yield a relationship between the system head and the flow rate. A plot of the flow rate vs. the system head is known as the *system head curve*.

In pump design, our fundamental task is to determine the flow rate that a given pump can accommodate for a required head difference. To accomplish this task, we can pair the system head curve with the specifications for our pump model—i.e. our pump curve that relates the discharge that can be supported for a given head differential.

Pump curves: head vs. discharge

Manufacturers provide *pump curves* that can be used to determine the pump's applicability for different use cases. The most fundamental pump curve is the head vs. discharge curve, which charts the head imparted by the pump vs. the flow rate through the pump. These pumps may be provided in graphical form:

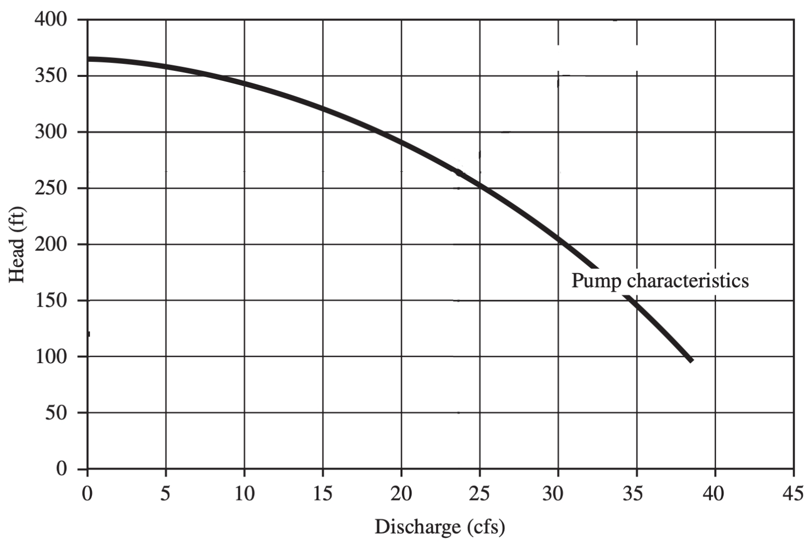


Figure 15: Example pump curve.

Or in an equivalent tabular form:

Table 6: Example pump curve in tabular form.

Q $[cfs]$	H_p $[ft]$
0.0	363.0
5.5	357.6
11.0	341.2
16.5	314.2
22.0	272.9
27.5	226.9
33.0	167.0
38.5	96.2

Often these pump curves may be approximated using empirical equations, such as the following quadratic approximation:

Quadratic approximation of pump curve

$$H_p(Q) \approx -aQ^2 - bQ + c \tag{223}$$

If we want to determine the suitability of a given pump for our application, we need to compare the pump curve against the head and flow rate required by our pipe system of interest. Recall from our two-reservoir example that the desired system head required from the pump is given by:

$$H_{sys} = H_{stat} + h_L \tag{224}$$

Moreover, we know that the head loss depends on the flow rate through the system with the general form:

$$h_L = KQ^m \tag{225}$$

Where K and m depend on our chosen friction equation (e.g. Darcy-Weisbach, Hazen-Williams, Chezy-Manning). Thus, we can also write our desired pump head (i.e. our system head H_{sys}) as a function of the flow rate Q:

$$H_{sys}(Q) = H_{stat} + KQ^m \tag{226}$$

It follows that if we compare the pump curve (representing the head-discharge relationship for the pump) against the system curve (representing the head-discharge relationship for the system), we can determine the *operating point* (H_p^*, Q^*) at which the pump will operate. Using the quadratic form of the pump curve, for example, we get:

$$H_p(Q) = H_{sys}(Q) \tag{227}$$

$$-aQ^2 - bQ + c = H_{stat} + KQ^m \tag{228}$$

$$0 = KQ^m + aQ^2 + bQ + c + H_{sys} \tag{229}$$

Or, shown graphically, we are interested in determining the point at which the pump curve and system curve intersect:

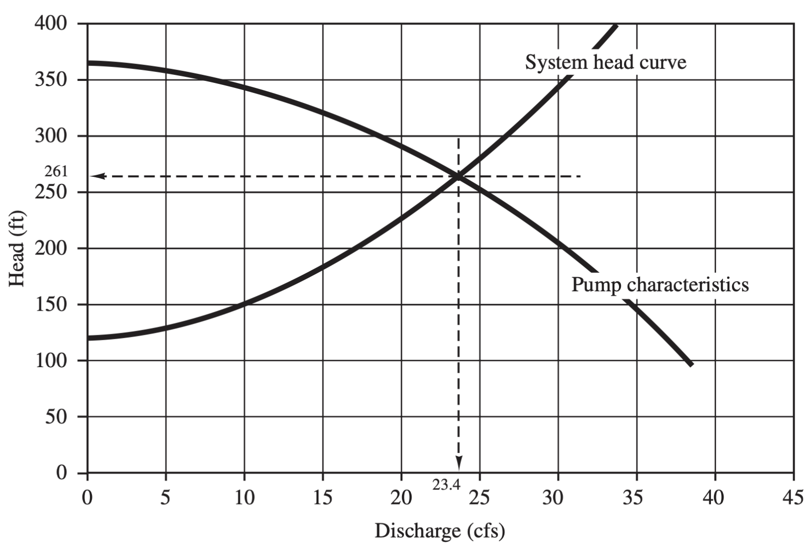
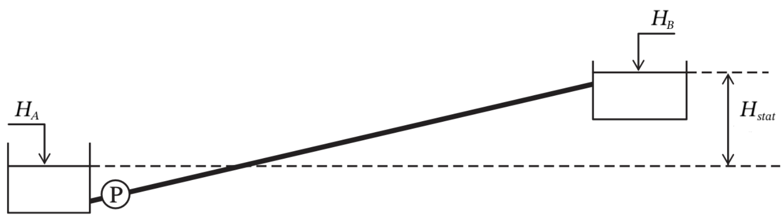


Figure 16: Finding the pump operating points consists of finding the intersection of the pump curve and the system head curve.

EXAMPLE 3.1

Problem: Consider the pump-pipeline system shown below. The reservoir water surface elevations are $H_A = 100\ [ft]$ and $H_B = 220\ [ft]$. The 2 [ft] diameter pipe connecting the two reservoirs has a length of 12,800 [ft] and a Hazen-Williams coefficient of $C_{hw} = 100$. Determine the discharge in the pipeline, the velocity of flow, and the energy grade line.



Solution: For this system, the static head $H_{stat} = H_B - H_A = 220 - 100\ [ft] = 120\ [ft]$. Friction losses can be determined using the Hazen-Williams formula:

$$h_L = KQ^{1.85} \tag{230}$$

Computing K:

$$K = \frac{4.73\Delta x}{D^{4.87}C_{hw}^{1.85}} = \frac{4.73(12,800)}{(2)^{4.87}(100)^{1.85}} = 0.413\ [s^{1.85}/ft^{4.55}] \tag{231}$$

The system head curve can thus be computed for each value of Q according to the formula:

$$H_{sys}(Q) = 120 + 0.413Q^{1.85} \tag{232}$$

Similarly, using the curve-fitting functionality in Excel, we can determine a function to approximate the pump curve:

$$H_p(Q) = -0.177Q^2 - 0.1101Q + 300.31 \tag{233}$$

The problem can now be solved by determining the operating point where the two curves intersect. This can be done graphically (as shown above) or numerically (using solver or goal-seek in Excel). If we wish to compute the operating point numerically, we first set the two equations equal to each other and rearrange to yield zero:

$$120 + 0.413Q^{1.85} = -0.177Q^2 - 0.1101Q + 300.31 \tag{234}$$
$$0 = 0.177Q^2 + 0.413Q^{1.85} + 0.1101Q - 180.31 \tag{235}$$

Performing this analysis, we find that the discharge that meets the required head is:

$$\boxed{Q = 20\ [cfs]} \tag{236}$$

And the head difference across the pump at this flow rate is:

$$H_p = 120 + 0.413 \cdot (20)^{1.85} \tag{237}$$

$$\boxed{H_p = 226\ [ft]} \tag{238}$$

EXAMPLE 3.2

Problem: For the same system shown above, it is desired to supply flow rate at 12 [cfs]. What is the head that the pump will be able to provide?

Solution: Using Excel to fit a quadratic function to the pump curve, we have:

$$H_p(Q) = -0.177Q^2 - 0.1101Q + 300.31 \tag{239}$$

Substituting our value for Q:

$$H_p(12) = -0.177(12)^2 - 0.1101(12) + 300.31 \tag{240}$$

$$\boxed{H_p = 273.5\ [m]} \tag{241}$$

EXAMPLE 3.3

Example: For the same system shown above, a project requires a pump head of 282 [ft] at a flow rate of 30 [cfs]. Will this pump work for this application?

Solution: Using our quadratic approximation to the pump curve, we have:

$$H_p(Q) = -0.177Q^2 - 0.1101Q + 300.31 \tag{242}$$

Substituting our value for Q:

$$H_p(30) = -0.177(30)^2 - 0.1101(30) + 300.31 \tag{243}$$

$$H_p = 138\ [m] \tag{244}$$

Thus, this pump will not be able to provide the required flow rate at this pump head.

Pump power and efficiency

In addition to designing for a given head-discharge operating point, it is often important to consider pump efficiency and pump power limitations. Pump power limitations are often expressed in terms of *brake horsepower*, which describes the power input to the pump.

The following graph shows the pump curves for pump head, efficiency, and brake horsepower as a function of flow rate for a particular pump model:

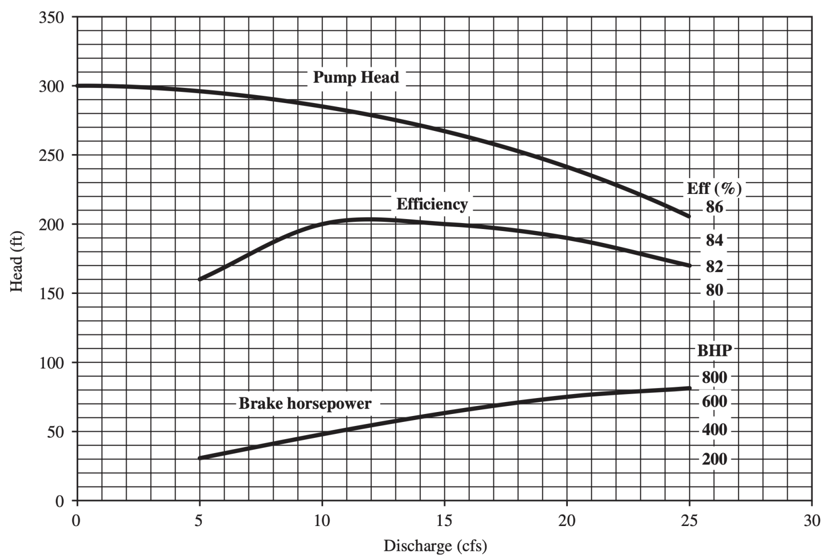


Figure 17: Pump curves for head-discharge, head-efficiency, and head-horsepower relations.

Pump power and efficiency

The *input power*, P_i, to a pump is equal to the angular velocity of the impeller ω times the torque T applied by the motor.

Input power to pump

$$\boxed{P_i = \omega T = \frac{2\pi N}{60} T} \qquad (245)$$

Where N is the rotational speed in revolutions per minute (rpm).

The *output power*, P_o, of the pump can be expressed as follows:

Output power of pump

$$\boxed{P_o = \rho g H_p Q} \quad (246)$$

Where ρ is the density of water, g is the acceleration due to gravity, H_p is the head added by the pump and Q is the flow rate through the pump.

Where does this expression come from? Recall from earlier that head refers to the energy per unit weight for a control volume of fluid. Thus, the head imparted by the pump is equal to the work performed per unit weight of water. Multiplying this work by the specific weight $\gamma = \rho g$ yields the work performed per unit of volume of water. Thus, the meaning of the above equation is as follows:

$$P_o = [\rho g H_p] \cdot [Q] \quad (247)$$

$$\begin{aligned} P_o = &[\text{Work performed per unit volume of water}] \\ &\cdot[\text{Volumetric flow rate through control volume}] \end{aligned} \quad (248)$$

The *pump efficiency* is defined as the ratio of output power to input power, and depends on the design of the pump (e.g. vanes, housing, conditions under which pump operates).

Pump efficiency

$$\boxed{\eta_p = \frac{P_o}{P_i}} \quad (249)$$

The *motor efficiency* is defined as the ratio of power output applied to the pump from the motor:

Motor efficiency

$$\boxed{\eta_m = \frac{P_i}{P_m}} \quad (250)$$

The overall efficiency of the pump is equal to the product of the pump efficiency and motor efficiency:

Overall efficiency

$$\boxed{\eta = \eta_p \eta_m = \frac{P_o}{P_m}} \quad (251)$$

EXAMPLE 3.4

Problem: A 10-ft diameter propeller pump is installed to deliver a large quantity of water between two reservoirs with a water elevation difference of 8.5 ft. The shaft power supplied to the pump is 2000 hp. The pump operates at 80% efficiency. Determine the discharge rate and the pressure just downstream of the pump if the pressure just upstream is 12 psi. Assume short pipe and friction losses are negligible.

Solution: Applying an energy balance between reservoir 1 and reservoir 2:

$$z_1 + H_p = z_2 + K_{en}\frac{u^2}{2g} + K_{ex}\frac{u^2}{2g} \tag{252}$$

$$H_p = (z_2 - z_1) + \frac{u^2}{2g}(K_{en} + K_{ex}) \tag{253}$$

$$H_p = (z_2 - z_1) + \frac{Q^2}{2gA^2}(K_{en} + K_{ex}) \tag{254}$$

From the pump power equation, we have:

$$P_o = \rho g Q H_p \tag{255}$$

$$H_p = \frac{P_o}{\rho g Q} \tag{256}$$

Moreover, we have that the pump output power is equal to the efficiency times the input power:

$$P_o = \eta P_i \tag{257}$$

Thus:

$$H_p = \frac{\eta P_i}{\rho g Q} \tag{258}$$

Returning to our energy balance:

$$H_p = (z_2 - z_1) + \frac{Q^2}{2gA^2}(K_{en} + K_{ex}) \tag{259}$$

$$\frac{\eta P_i}{\rho g Q} = (z_2 - z_1) + \frac{Q^2}{2gA^2}(K_{en} + K_{ex}) \tag{260}$$

$$0 = (z_2 - z_1) + \frac{Q^2}{2gA^2}(K_{en} + K_{ex}) - \frac{\eta P_i}{\rho g Q} \tag{261}$$

Multiplying through by Q to yield a polynomial:

$$0 = \frac{(K_{en} + K_{ex})}{2gA^2}Q^3 + (z_2 - z_1)Q - \frac{\eta P_i}{\gamma} \tag{262}$$

Note that:

$$A = \pi\frac{D^2}{4} = 78.5\ [ft^2] \tag{263}$$

$$P_i = 2000\ [hp] \cdot \frac{550\ [ft - lb/s]}{1\ [hp]} = 1.1 \times 10^6 [ft - lb/s] \tag{264}$$

$$\begin{aligned} 0 = &\frac{(0.5 + 1)}{2(32.2\ [ft/s^2])(78.5\ [ft^2])^2}Q^3 \\ &+(8.5\ [ft])Q - \frac{0.8(1.1 \times 10^6[ft - lb/s])}{62.4\ [lb/ft^3]} \end{aligned} \tag{265}$$

$$0 = (3.77 \times 10^{-6})Q^3 + (8.5)Q - 14103 \tag{266}$$

Finding the roots of this polynomial, we find that the only real-valued root is:

$$\boxed{Q = 1088\ [cfs]} \tag{267}$$

Applying an energy balance from the upstream end of the pump to the downstream end, we have:

$$\frac{p_u}{\gamma} + z_u + \frac{u_u^2}{2g} + H_p = \frac{p_d}{\gamma} + z_d + \frac{u_d^2}{2g} \tag{268}$$

Because the inlet and outlet pipe are the same diameter, the velocity head is equal on both sides of the pump. Moreover, the upstream and downstream ends of the pump are at roughly the same elevation. Thus:

$$\frac{p_u}{\gamma} + H_p = \frac{p_d}{\gamma} \tag{269}$$

$$p_d = p_u + \gamma H_p \tag{270}$$

From before, we have that the pump head can be expressed as:

$$p_d = p_u + \gamma \frac{\eta P_i}{\gamma Q} \tag{271}$$

$$p_d = p_u + \frac{\eta P_i}{Q} \tag{272}$$

$$p_d = \frac{12\ [lb]}{1[in^2]} + \frac{0.8(1.1 \times 10^6\ [ft - lb/s])}{1088\ [ft^3/s]} \tag{273}$$

$$p_d = 12\ [lb/in^2] + (809\ [lb/ft^2])\left(\frac{1\ [ft]}{12[in]}\right)^2 \tag{274}$$

Thus, the pressure directly downstream of the pump is:

$$\boxed{p_d = 17.6\ [lb/in^2]} \tag{275}$$

Variable-speed pumps

Some pumps can be operated at different speeds corresponding to different power output levels. Flow, head, and brake horsepower all vary with rotational speed. The change in these quantities can be related to changes in speed by *pump affinity laws*:

Pump affinity law: flow rate

$$\boxed{\frac{Q_1}{Q_2} = \left(\frac{N_1}{N_2}\right)} \tag{276}$$

Pump affinity law: pump head

$$\boxed{\frac{H_1}{H_2} = \left(\frac{N_1}{N_2}\right)^2} \tag{277}$$

Pump affinity law: pump power

$$\boxed{\frac{P_1}{P_2} = \left(\frac{N_1}{N_2}\right)^3} \tag{278}$$

Example 3.5

Problem: The table shown below corresponds to the pump operating at 2000 rpm. Compute the pump curve for the pump operating at 2200 rpm.

$Q(2000\ [rpm])\ [cfs]$	$H_p(2000\ [rpm])$
0	300.0
5	295.5
10	282.0
15	259.5
20	225.5
25	187.5
30	138.0
35	79.5

Solution: Using the affinity laws, we have that:

$$\frac{N_2}{N_1} = 1.1 \tag{279}$$

$$\left(\frac{N_2}{N_1}\right)^2 = 1.21 \tag{280}$$

Thus, we can compute the new head-flow rate curve by multiplying the flow rates and heads at 2000 rpm by this factor:

$$Q_1 = \frac{N_2}{N_1} Q_2 = 1.1 \cdot Q_2 \tag{281}$$

$$H_1 = \left(\frac{N_2}{N_1}\right)^2 H_2 = 1.21 \cdot H_2 \tag{282}$$

$Q(2000\ [rpm])\ [cfs]$	$H_p(2000\ [rpm])\ [ft]$	$Q(2200\ [rpm])$	$H_p(2200\ [rpm])\ [ft]$
0	300.0	0.0	363
5	295.5	5.5	358
10	282.0	11.0	341
15	259.5	16.5	314
20	225.5	22.0	273
25	187.5	27.5	227
30	138.0	33.0	167
35	79.5	38.5	96

Pumps in parallel and series

Pumps in parallel

Pumps can be operated in parallel to deliver more flow at the same head as a single pump.

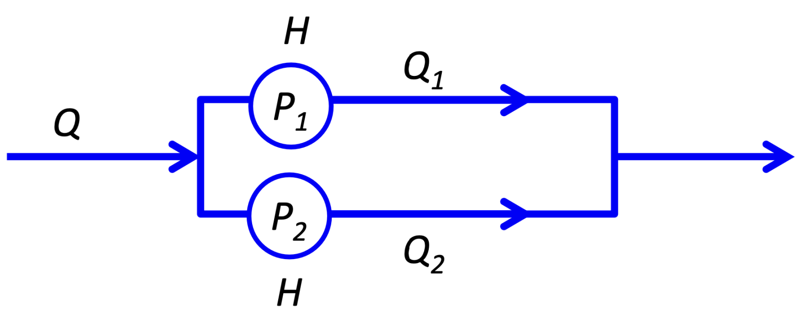


Figure 18: Pumps in parallel.

For a set of N pumps operated in parallel, the flow rate is the sum of the flow rates through the individual pumps:

$$Q_{total} = \sum_{i=1}^{N} Q_i \tag{283}$$

Given that all pumps operate at the same pump head, the total pump head is equal to the pump head of each individual pump:

$$H_{total} = H_i \;\; \forall \;\; i \tag{284}$$

For pumps in parallel with pump curves given by $H_1(Q)$ and $H_2(Q)$, we can obtain an equivalent combined pump curve by adding together the discharges for each pump over the range for which the pump heads are defined for both pumps:

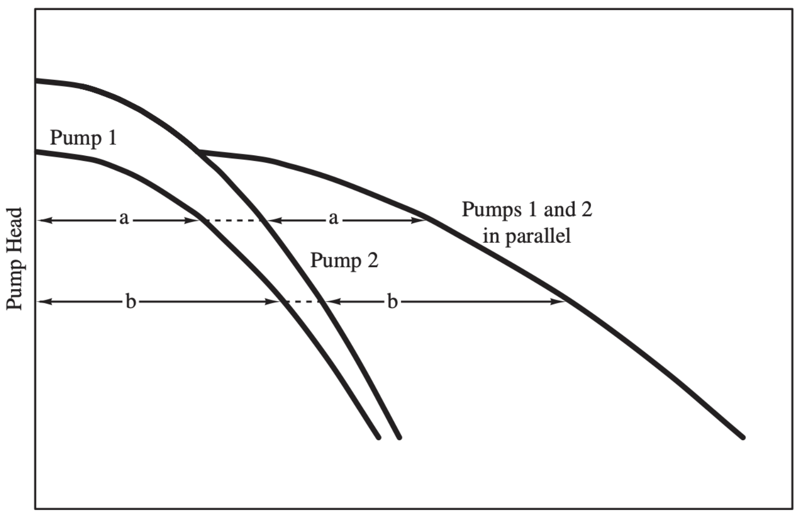


Figure 19: Equivalent combined pump curve for two pumps acting in parallel

Pumps in series

Pumps can be operated in series to deliver greater pump head at the same flow rate.

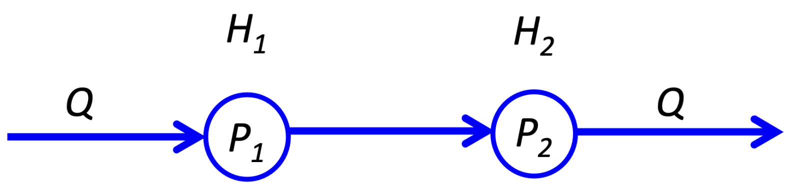


Figure 20: Pumps in series.

For a set of N pumps operated in series, the flow rate through all pumps is the same:

$$Q_i = Q_j \ \forall \ i, j \tag{285}$$

Total total head imparted to the fluid is the sum of the individual pump heads:

$$H_{total} = \sum_{i=1}^{N} H_i \tag{286}$$

For pumps in series with pump curves given by $H_1(Q)$ and $H_2(Q)$, we can obtain an equivalent combined pump curve by adding together the heads for each pump:

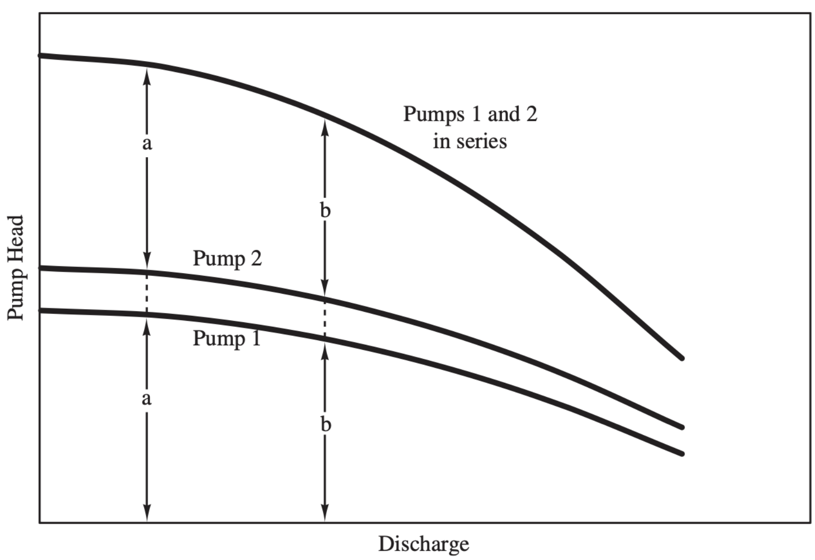


Figure 21: Equivalent combined pump curve for two pumps acting in series.

Part V

Open channel flow

Overview of open channel flow

Differences between pipe and open channel flow

Before we begin our study of open channel flow, it is helpful to note the ways in which open channel flow differs from pressurized pipe flow. In many ways, these two types of flow are similar. We will analyze these systems using the same basic tools, including conservation of mass, conservation of momentum, and conservation of energy. We will even apply many of the same techniques for resolving energy losses, such as the Chezy-Manning equation for head losses due to friction. However, pipe flow and open channel flow differ in a few important respects. For the purposes of this class, the most important differences are as follows: (i) for open channel flow, the mass of water in a given control volume may change over time, allowing for the progression of waves under unsteady conditions, (ii) for open channel flow, the energy balance is affected by changes in the hydraulic geometry, which is in turn varies with the depth of water in the channel, and (iii) for open channel flow, pressure forces acting upon a control volume of water are generally represented by the hydrostatic pressure, which is proportional to the water depth.

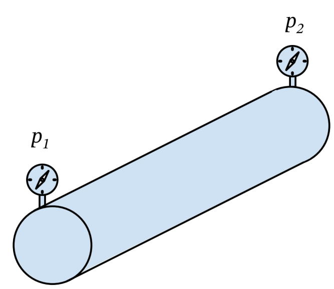


Figure 22: Pressurized pipe flow.

Differences between pipe and open channel flow: control volumes

For pressurized pipe flow, fluid is assumed to fill the entire conduit such that flow boundaries are fixed by the conduit geometry. For the case of unsteady flow, this means that the mass of water within a control volume of pressurized pipe remains constant in time, and the flow into the pipe is always equal to the flow out (assuming incompressible flow). For open channels, this condition does not apply, and the volume of water within a control volume of open channel may change over time, meaning that the flow into the control volume is not necessarily equal to the flow out of the control volume under unsteady conditions. An example of highly unsteady flow in an open channel system might include the propagation of a flood wave resulting from a dam release or intense rainfall.

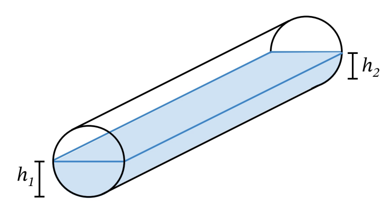


Figure 23: Open channel flow.

Differences between pipe and open channel flow: hydraulic geometry

For pipes, we have so far assumed that shape is circular, and that the cross-sectional area of flow does not vary with the flow rate itself. Open channels, on the other hand, feature a variety of cross sectional shapes. These include *prismatic channels*, which may be rectangular, square, triangular, trapezoidal, or semicircular in cross section. Open channels may also feature *irregular geometries*. Generally, constructed channels will be prismatic, whereas natural channels will be irregular.

Moreover, for open channels, the hydraulic geometry varies with the depth of water. Thus, to characterize flow in open channels, we typically define the following quantities as a function of the water depth:

Cross sectional area: Denoted as $A(h)$, this quantity represents the cross sectional area as a function of water depth.

Wetted perimeter: Denoted as $P(h)$, this quantity represents the perimeter of the channel bed that is in contact with the water.

Top width: Denoted as $T(h)$, the top width is the width of the water surface over the cross section. It is also equal to the rate of change of the cross sectional area with respect to water depth: $T(h) = dA(h)/dh$.

Hydraulic radius: Denoted as $R(h)$, the hydraulic radius is the area of the cross section divided by the wetted perimeter: $R(h) = A(h)/P(h)$

EXAMPLE 4.1

Problem: Compute the cross-sectional area of flow A(h) and the wetted perimeter P(h) as a function of depth for a rectangular channel with bottom width B:

Solution: For the cross sectional area, we have:

$$\boxed{A(h) = Bh} \tag{287}$$

For the wetted perimeter, we have:

$$\boxed{P(h) = B + 2h} \tag{288}$$

For the top width, we have:

$$\boxed{T(h) = B} \tag{289}$$

Example 4.2

Problem: Compute the cross-sectional area of flow A(h) and the wetted perimeter P(h) as a function of depth for a triangular channel with side slope m (horizontal to vertical):

Solution: For the area of a right triangle corresponding to half the channel, we have:

$$\frac{1}{2}A(h) = \frac{1}{2}bh = \frac{1}{2}(mh)h \tag{290}$$

Thus:

$$\boxed{A(h) = mh^2} \tag{291}$$

The wetted perimeter is equal to twice the length of the hypotenuse:

$$P(h) = 2\sqrt{h^2 + b^2} = 2\sqrt{h^2 + m^2h^2} \tag{292}$$

Thus:

$$\boxed{P(h) = 2h\sqrt{1 + m^2}} \tag{293}$$

For the top width, we have that the top width is equal to twice the base of the right triangle corresponding to half the channel:

$$T(h) = 2b \tag{294}$$

Thus:

$$\boxed{T(h) = 2mh} \tag{295}$$

Differences between pipe and open channel flow: energy and forces

Pipe and open channel flow also differ with respect to the forces that drive flow. Comparing the forces acting on a control volume of water in a pipe versus a control volume of water in an open channel, we see that the forces are mostly the same. Both control volumes are acted upon by the force of gravity, as well as a frictional force. However, the pressure force differs between the two cases. For pipe flow, the pressures applied at the boundaries of the control volume depend on the specific conditions upstream and downstream (including the presence of pumps, reservoirs, and frictional losses). However, for open channel flow, the pressure is generally hydrostatic:

$$p = \rho g h = \gamma h \tag{296}$$

Where $h = p/\gamma$ is the depth of water above the channel bottom. Thus, replacing the pressure term in the momentum equation with γh we get:

Conservation of momentum for fluid flow in open channels in differential form

$$\boxed{\frac{\partial Q}{\partial t} + \frac{\partial (Qu)}{\partial x} + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f\right) = 0} \tag{297}$$

And similarly, replacing the pressure term in the energy equation with γh, we get:

Energy equation for open channel flow

$$\boxed{\frac{u_1^2}{2g} + h_1 + z_1 = \frac{u_2^2}{2g} + h_2 + z_2 + h_L} \tag{298}$$

Where z is the elevation of the channel bottom, h is the depth of water, u is the mean velocity of flow, and h_L is the head losses. For a given point in the channel, we can define the following quantities:

- *Total head:* $h + z + \frac{u^2}{2g}$
- *Piezometric head:* $h + z$
- *Velocity head:* $\frac{u^2}{2g}$

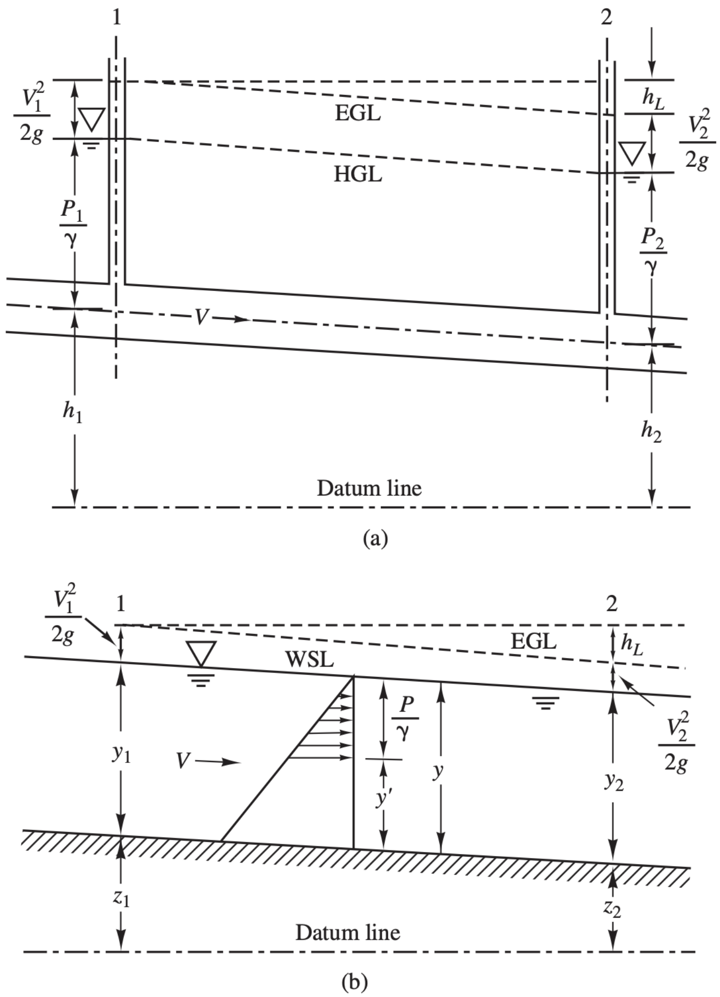


Figure 24: Energy characteristics of pipe vs. open channel flow.

Temporal classification of open channel flow

Temporally, open channel flow can be grouped into two classifications:

1. **Steady flow:** The depth and velocity of flow do not vary with time.

2. **Unsteady flow:** The depth and velocity of flow do vary with time.

Classification of steady-state open channel flow

Steady-state open channel flow is divided into three general categories:

1. **Uniform flow:** The water depth remains the same throughout the length of the channel at a given time, or equivalently, the water surface is parallel with the channel bottom. ($\frac{dh}{dx} = 0$).

2. **Gradually-varied flow:** Water depth or velocity vary gradually along the length of the length of channel ($|\frac{dh}{dx}| << 1$).

3. **Rapidly-varied flow:** Water depth or velocity vary abruptly along a short length of channel ($\frac{dh}{dx} \approx \pm 1$). Often, the pressure distribution will be non-hydrostatic.

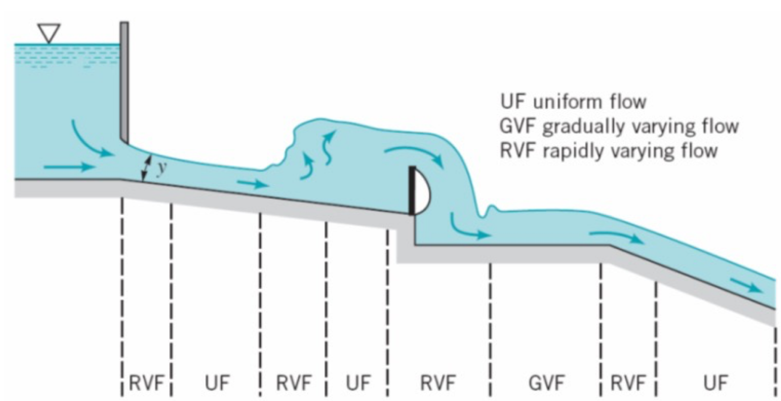


Figure 25: Gradually-varied, rapidly-varied, and uniform flow.

Uniform flow

Uniform flow occurs when the depth and velocity of flow remain constant over the length of the channel. This case represents one of the most common design conditions for hydraulic infrastructure. In this section, we will derive an equation that relates the discharge and depth of flow in an open channel under uniform flow conditions. Starting with the momentum equation:

$$\frac{\partial Q}{\partial t} + \frac{\partial (Qu)}{\partial x} + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f\right) = 0 \tag{299}$$

By the product rule, the momentum flux term can be partitioned:

$$\frac{\partial Q}{\partial t} + u\frac{\partial Q}{\partial x} + Q\frac{\partial u}{\partial x} + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f\right) = 0 \tag{300}$$

For uniform flow, we have that the system is at steady state. We also have that the depth, velocity, and discharge remain constant throughout the length of the channel:

$$\cancelto{0}{\frac{\partial Q}{\partial t}} + u\cancelto{0}{\frac{\partial Q}{\partial x}} + Q\cancelto{0}{\frac{\partial u}{\partial x}} + gA\left(\cancelto{0}{\frac{\partial h}{\partial x}} - S_0 + S_f\right) = 0 \tag{301}$$

In other words, *the friction force exactly balances the gravitational force*. And thus we have:

$$0 = S_0 - S_f \tag{302}$$

Recall that the friction slope is defined as:

$$S_f = \frac{\tau P}{gA\rho} = \frac{\tau}{g\rho R} \tag{303}$$

Where R is the hydraulic radius, which is equal to the cross-sectional area of flow divided by the wetted perimeter:

$$R = \frac{A}{P} \tag{304}$$

Thus, under normal flow conditions where $S_f = S_0$:

$$\tau = g\rho R S_f = g\rho R S_0 \tag{305}$$

From Newton's drag law, the shear stress for fully turbulent flow can be expressed as a function of density, velocity, and a resistance coefficient C_f:

$$\tau = C_f \rho \frac{u^2}{2} \tag{306}$$

Equating the two expressions for the shear stress:

$$g\rho R S_0 = C_f \rho \frac{u^2}{2} \tag{307}$$

Solving for the velocity gives:

$$u = \sqrt{\frac{2g}{C_f}} \sqrt{RS_0} \tag{308}$$

Let's define $C = \sqrt{2g/C_f}$. Then the equation above can be simplified to the well-known *Chezy equation*:

$$u = C\sqrt{RS_0} \tag{309}$$

Where C is referred to as the *Chezy coefficient*. An empirical value for C was given by Manning, based on a statistical average of different empirical formulations:

$$C = \frac{\phi}{n} R^{1/6} \tag{310}$$

Where n is the Manning roughness coefficient, and ϕ is a constant that depends on the unit system employed. Substituting this formulation for C into the Chezy equation yields the so-called *Manning's equation*:

$$u = \frac{\phi}{n} R^{2/3} S_0^{1/2} \tag{311}$$

More typically, Manning's equation is written in terms of the discharge by multiplying by the cross-sectional flow area.

Manning's equation for uniform flow

$$\boxed{Q = \frac{\phi}{n} A R^{2/3} S_0^{1/2}} \tag{312}$$

The constant ϕ changes depending on whether metric or U.S. customary units are used:

- For Q in cubic meters per second (SI units), $\phi = 1$.
- For Q in cubic feet per second (U.S. customary units), $\phi = 1.49$.

EXAMPLE 4.3

Problem: An 8-ft wide rectangular channel with a bed slope of 0.0004 ft/ft has a depth of flow of 2 ft. Assuming steady uniform flow, determine the discharge in the channel. Assume a Manning roughness coefficient of $n = 0.015$.

Solution: Applying Mannings equation:

$$Q = \frac{\phi}{n} A R^{2/3} S_0^{1/2} \tag{313}$$

$$= \frac{1.49}{0.015}(8)(2)\left[\frac{(8)(2)}{8+2(2)}\right]^{2/3}(0.0004)^{1/2} \tag{314}$$

Thus:

$$\boxed{Q = 38.5\ [ft^3/s]} \tag{315}$$

EXAMPLE 4.4

Problem: For the channel described in the previous example, determine the normal depth if the flow rate is 100 cfs.

Solution: Applying Mannings equation:

$$Q = \frac{\phi}{n} A R^{2/3} S_0^{1/2} \tag{316}$$

$$= \frac{1.49}{n} S_0^{1/2} \frac{(Bh)^{5/3}}{(B+2h)^{2/3}} \tag{317}$$

This equation can be solved numerically for h. You can use a graphing calculator to find the root, for example. We'll do it using Python.

```
# Import modules
import numpy as np
import scipy.optimize

# Define a function for which to find the root
def normal_flow(h, phi=1.49, Q=100, B=8, S_0=0.0004, n=0.015):
   Q_c = (phi * np.sqrt(S_0) / n) * (B * h)**(5/3) / (B + 2 *
       h)**(2/3)
   return Q_c - Q

# Find the root using bisection
result = scipy.optimize.root_scalar(normal_flow, method='bisect',
    bracket=[0., 10.])
```

```
# Print the root
print(result.root)
```

The resulting root gives the normal depth:

$$\boxed{h = 3.97\ [ft]} \tag{318}$$

EXAMPLE 4.5

Problem: Consider a trapezoidal channel with bottom width of $B = 25\ [ft]$ and side slopes 3:2 horizontal to vertical. The Manning's roughness is $n = 0.017$ and the bottom slope is $S_o = 0.00088$. How deep does the channel need to be to accommodate a discharge of $Q = 2,000\ [cfs]$?

Solution: Applying Mannings equation:

$$Q = \frac{\phi}{n} AR^{2/3} S_0^{1/2} \tag{319}$$

$$= \frac{1.49}{n} S_0^{1/2} \frac{((B+mh)h)^{5/3}}{(B+2h\sqrt{1+m^2})^{2/3}} \tag{320}$$

Again, this equation can be solved numerically. You can use a graphing calculator to find the root, for example. We'll do it using Python.

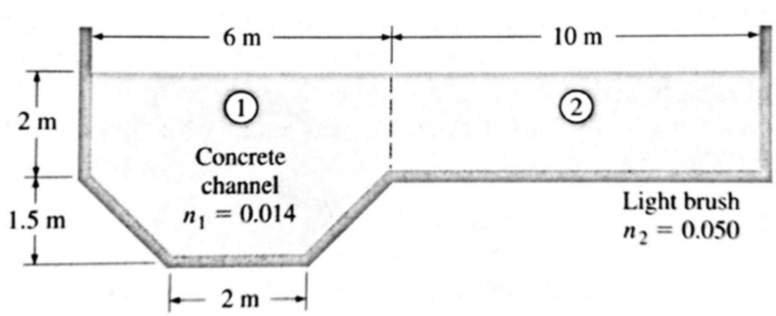
```
# Import modules
import numpy as np
import scipy.optimize

# Define a function for which to find the root
def normal_flow(h, phi=1.49, Q=2000, B=25, m=1.5, S_0=0.00088,
    n=0.017):
   Q_c = (phi * np.sqrt(S_0) / n) * ((B + m * h) * h)**(5/3) / (B
       + 2 * h * np.sqrt(1 + m**2))**(2/3)
   return Q_c - Q

# Find the root using bisection
result = scipy.optimize.root_scalar(normal_flow, method='bisect',
    bracket=[0., 100.])

# Print the root
print(result.root)
```

The resulting root gives the normal depth:

$$\boxed{h = 7.25\ [ft]} \tag{321}$$

EXAMPLE 4.6

Problem: Consider the following irregular channel section with a bottom slope of $S_o = 0.004$:



1. Compute the flow rate through the total cross section.
2. Compute an equivalent single Manning's roughness factor for the entire channel.

Solution: Starting with the first subproblem:

1. Apply Mannings equation individually to each section. First, let's compute the area and wetted perimeter of each section:

For the first section, we have:

$$B_1 = 2\ [m] \tag{322}$$
$$B_2 = 6\ [m] \tag{323}$$
$$h_1 = 1.5\ [m] \tag{324}$$
$$h_2 = 2\ [m] \tag{325}$$
$$m = 2/1.5 = 1.33 \tag{326}$$

Thus, for the area, we have:

$$A_1 = (B_1 + mh_1)h_1 + B_2h_2 \tag{327}$$
$$= (2 + 1.33 \cdot 1.5) \cdot 1.5 + 2 \cdot 6\ [m] \tag{328}$$
$$= 18\ [m^2] \tag{329}$$

For the wetted perimeter we have:

$$P_1 = h_2 + B_1 + 2\sqrt{m^2 h_1^2 + h_1^2} \tag{330}$$
$$= 2 + 2 + 2 \cdot 1.5\sqrt{1.33^2 + 1} \tag{331}$$
$$= 9\ [m] \tag{332}$$

For the second section we have:

$$B_3 = 10\ [m] \tag{333}$$
$$h_3 = 2\ [m] \tag{334}$$

Thus, for the area we have:

$$A_2 = B_3 h_2 = 10 \cdot 2\ [m] \tag{335}$$
$$= 20\ [m^2] \tag{336}$$

For the wetted perimeter we have:

$$P_2 = B_2 + h_3 = 10 + 2\ [m] \tag{337}$$
$$= 12\ [m] \tag{338}$$

The total discharge is given by:

$$Q = Q_1 + Q_2 = \frac{1}{n_1}\frac{A_1^{5/3}}{P_1^{2/3}} S_o^{1/2} + \frac{1}{n_2}\frac{A_2^{5/3}}{P_2^{2/3}} S_o^{1/2} \tag{339}$$
$$= \frac{1}{0.014}\frac{(18)^{5/3}}{(9)^{2/3}}(0.004)^{1/2} + \frac{1}{0.050}\frac{(20)^{5/3}}{(12)^{2/3}}(0.004)^{1/2} \tag{340}$$
$$= 129 + 36\ [cms] \tag{341}$$

Thus, the total discharge is:

$$\boxed{Q = 165\ [cms]} \tag{342}$$

2. Let $A = A_1 + A_2$ and $P = P_1 + P_2$. Then we have:

$$Q = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}} S_o^{1/2} \tag{343}$$

Substituting numeric values:

$$165 = \frac{1}{n}\frac{(18 + 20)^{5/3}}{(9 + 12)^{2/3}}(0.004)^{1/2} \tag{344}$$
$$n = \frac{1}{165}\frac{(38)^{5/3}}{(21)^{2/3}}(0.004)^{1/2} \tag{345}$$

Thus, the equivalent Manning's roughness is:

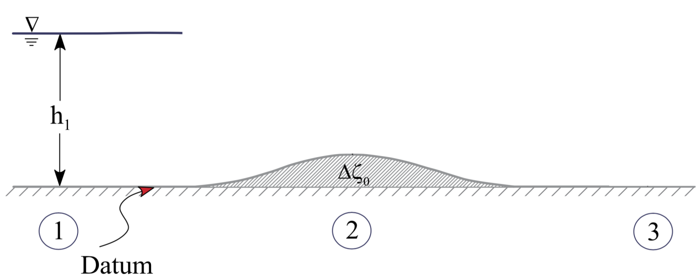
$$\boxed{n = 0.021} \tag{346}$$

Energy principles in open-channel flow

Gradual transitions in flow

Thus far, we have considered the case of uniform flow, which occurs when the depth and velocity of water do not change over the length of channel. This condition typically applies when the open channel maintains a constant geometry and bottom slope for an extended length. But what happens when the channel geometry or bottom elevation change?

Consider flow in a nearly-horizontal open channel that encounters a smooth ridge at the channel bottom, such that the bottom elevation of the channel rises by $\Delta\zeta_0$. What happens to the depth of flow as it encounters the ridge? Does it go up? Does it go down? Or does it stay the same?



From before, we know that the energy equation for open channel flow is given by:

Energy equation for open channel flow

$$\boxed{\frac{u_1^2}{2g} + h_1 + z_1 = \frac{u_2^2}{2g} + h_2 + z_2 + h_L} \tag{347}$$

Let's try applying the energy equation and see if it is possible to deduce the depth of water at the crest of the ridge.

EXAMPLE 4.7

Problem: Consider a 3 $[m]$ wide rectangular channel. A smooth ridge is installed at the channel bottom. This ridge is 0.3 $[m]$ high at its peak. Upstream of the transition, the depth is equal to 3 $[m]$, and the velocity is measured to be 3 $[m/s]$. Assume no energy losses over the ridge or channel bottom. What is the absolute water level at the crest of the ridge?

Solution: First, let's compute the flow rate:

$$Q_1 = Q_2 = Q = u_1 A_1 = u_1 h_1 B_1 = 27\ [m^3/s] \tag{348}$$

$$h_1 + \frac{u_1^2}{2g} = h_2 + \Delta\zeta_0 + \frac{u_2^2}{2g} \tag{349}$$

$$h_1 + \frac{u_1^2}{2g} = h_2 + \Delta\zeta_0 + \frac{Q^2}{2gB^2h_2^2} \tag{350}$$

$$3 + \frac{(3)^2}{2(9.81)} = h_2 + 0.3 + \frac{27^2}{2g(3)^2h_2^2} \tag{351}$$

$$3.1587 = h_2 + \frac{4.1284}{h_2^2} \tag{352}$$

$$h_2^3 - 3.1587h_2^2 + 4.1284 = 0 \tag{353}$$

We can solve the roots of this polynomial using Python:

```python
# Import modules
import numpy as np

# Find the roots
roots = np.roots([1, -3.1587, 0, 4.1284])
print(roots)
```

Finding the roots of this cubic equation yields:

$$h_2 \in \{2.496, 1.659, -0.997\} \tag{354}$$

This solution implies that there are two non-negative depths that the water can take. So which one of these solutions is the correct depth? To answer this question, we first need to understand the conditions these two different solutions correspond to, and to accomplish this task, we first need to introduce the concepts of specific energy and subcritical and supercritical flow.

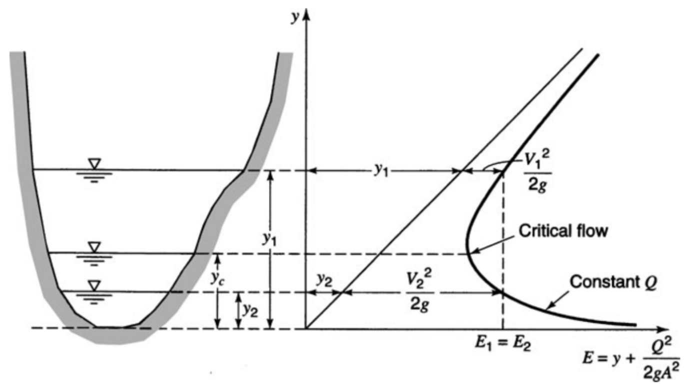
Specific energy and critical flow

We can define the specific energy as the total energy (head) above the channel bottom:

Specific energy

$$\boxed{E = h + \frac{u^2}{2g}} \tag{355}$$

You can think of the specific energy as the energy that can be "traded" between kinetic energy ($u^2/2g$) and potential energy (h). We can plot curves of the specific energy for a given Q.



For any fixed E, Q and A, there are two possible depths h (except at the inflection point of the curve). *Critical flow* occurs when the specific energy is minimum for a given discharge. We can find this minimum point by setting the derivative of the specific energy to zero with respect to h:

$$0 = \frac{dE}{dh} = \frac{d}{dh}\left[h + \frac{u^2}{2g}\right] \tag{356}$$

It can be seen that:

$$\frac{d}{dh}(h) = 1 \tag{357}$$

For the second term, we have:

$$\frac{d}{dh}\left(\frac{u^2}{2g}\right) = \frac{d}{dh}\left(\frac{Q^2}{2gA^2}\right) \tag{358}$$

Because Q is fixed (constant) for a given specific energy curve, we can move it outside the derivative:

$$\frac{d}{dh}\left(\frac{Q^2}{2gA^2}\right) = \frac{Q^2}{2g}\frac{d}{dh}\left(\frac{1}{A^2}\right) \tag{359}$$

By the chain rule:

$$\frac{d}{dh}(A^{-2}) = \frac{d(A^{-2})}{dA} \cdot \frac{dA}{dh} = -2A^{-3}\frac{dA}{dh} \tag{360}$$

Now, note that the change in cross sectional area with respect to the change in depth is simply the top width of flow, T:

$$\frac{dA}{dh} = T \tag{361}$$

Thus, going back to our original minimization problem, we have:

$$0 = \frac{d}{dh}\left[h + \frac{u^2}{2g}\right] \tag{362}$$

$$= 1 - \frac{Q^2 T}{gA^3} \tag{363}$$

$$= 1 - \frac{u^2 T}{gA} \tag{364}$$

Thus, critical flow conditions are characterized by the equality:

$$\boxed{\frac{u^2 T}{gA} = 1} \tag{365}$$

Condition for critical flow

Given that T and A are functions of depth, this equality gives us a one-to-one relationship between the flow Q and depth h. This expression is related to the *Froude Number*, F_r, which is 1 at critical flow.

$$\boxed{F_r = \frac{u\sqrt{T}}{\sqrt{gA}} \begin{cases} < 1 & \text{subcritical flow} \\ = 1 & \text{critical flow} \\ > 1 & \text{supercritical flow} \end{cases}} \tag{366}$$

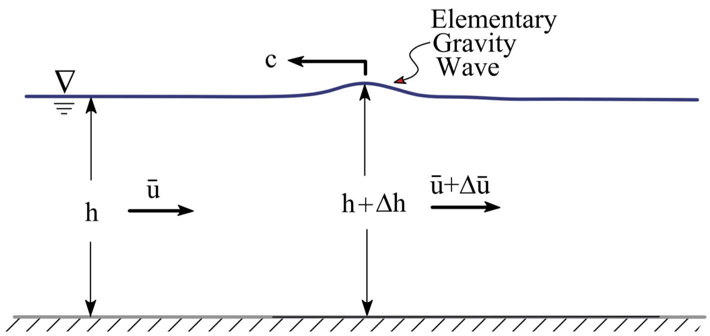
Froude number

Critical velocity and gravity wave speed

Let's talk about what the Froude number actually means. The Froude number characterizes the speed of water relative to the speed of a gravity wave produced by a disturbance in the free water surface. We will show that the celerity of a small gravity wave is equal to the velocity of flow corresponding to the critical depth:

$$\bar{u}_c = \sqrt{gh_c} \tag{367}$$

Consider a small surface gravity wave propagating with celerity (wave speed) c and an amplitude Δh in a rectangular channel of otherwise quiescient water at a constant depth h.



To simplify the analysis, we'll use a reference frame that travels with the wave, such that (in our reference frame) the wave remains stationary while the fluid moves to the right with velocity $\bar{u} = c$. The depth-averaged velocity under the stationary wave is accordingly modified by $\Delta\bar{u}$ to satisfy continuity, given that the depth there has been changed. Conservation of mass between the section of undisturbed flow and that of the gravity wave results in:

$$\bar{u}hB = (\bar{u} + \Delta\bar{u})(h + \Delta h)B \qquad (368)$$

$$\bar{u}h = (\bar{u} + \Delta\bar{u})(h + \Delta h) \qquad (369)$$

Expanding the term on the right-hand side, we obtain terms containing a product of the elementary quantities Δh and $\Delta\bar{u}$. For a "small" wave, the product of these terms ($\Delta\bar{u}\Delta h$) is negligible, and thus the previous equation can be simplified as follows:

$$\bar{u}h = \bar{u}h + \bar{u}\Delta h + \Delta\bar{u}h + \cancelto{0}{\Delta\bar{u}\Delta h} \qquad (370)$$

$$\bar{u}\Delta h + h\Delta\bar{u} = 0 \qquad (371)$$

Applying conservation of energy between the same two sections of the channel yields:

$$h + \frac{\bar{u}^2}{2g} = (h + \Delta h) + \frac{(\bar{u} + \Delta\bar{u})^2}{2g} \qquad (372)$$

Upon expanding, the higher-order term Δu^2 can be neglected. Thus, we obtain:

$$h + \frac{\bar{u}^2}{2g} = h + \Delta h + \frac{\bar{u}^2 + 2\bar{u}\Delta\bar{u} + \cancelto{0}{\Delta\bar{u}^2}}{2g} \qquad (373)$$

$$h - h = \Delta h + \frac{\bar{u}^2 - \bar{u}^2 + 2\bar{u}\Delta\bar{u}}{2g} \qquad (374)$$

$$\Delta h + \frac{1}{g}\bar{u}\Delta\bar{u} = 0 \qquad (375)$$

From our first expression we can now isolate $\Delta\bar{u}$:

$$\bar{u}\Delta h + h\Delta\bar{u} = 0 \tag{376}$$

$$\Delta\bar{u} = -\frac{\bar{u}\Delta h}{h} \tag{377}$$

Substituting this into the second expression:

$$\Delta h + \frac{1}{g}\bar{u}\Delta\bar{u} = 0 \tag{378}$$

$$\Delta h - \frac{\bar{u}^2\Delta h}{gh} = 0 \tag{379}$$

$$1 - \frac{\bar{u}^2}{gh} = 0 \tag{380}$$

$$\bar{u}^2 = gh \tag{381}$$

And thus, we find that the speed of the gravity wave is equal to:

Speed of a small gravity wave in a rectangular channel

$$\boxed{\bar{u} = \sqrt{gh}} \tag{382}$$

Now, let's return to the Froude number. Note that for a rectangular channel:

$$F_r = \frac{u\sqrt{T}}{\sqrt{gA}} = \frac{u\sqrt{B}}{\sqrt{gBh}} = \frac{u}{\sqrt{gh}} \tag{383}$$

Thus, the Froude number gives the ratio between the velocity of flow and the velocity of a gravity wave (disturbance) in the channel. If the Froude number is 1, that means that the gravity wave moves at the exact same speed as the flow in the channel, meaning that *disturbances cannot propagate upstream*.

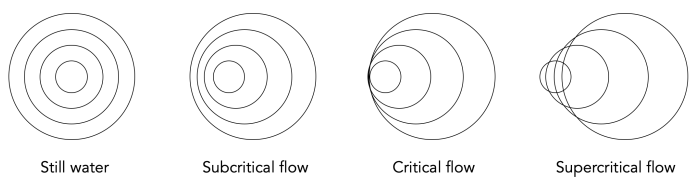
Subcritical, critical, and supercritical flow

Subcritical flow: Occurs when $F_r < 1$, and thus the water moves slower than a small gravity wave. Subcritical flow is associated with slow-moving flows on mild slopes such as backwaters from downstream impoundments.

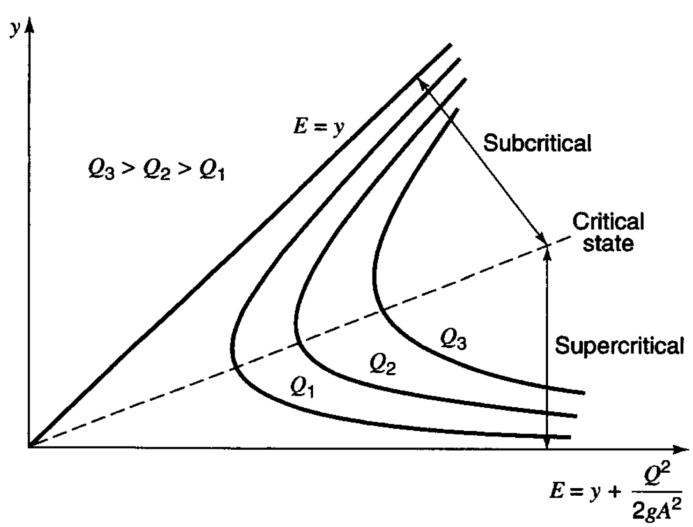
Supercritical flow: Occurs when $F_r > 1$, and thus the water moves faster than a small gravity wave. Supercritical flow is associated with rapid flows on steep slopes, such as the flow down dam spillways.

Perhaps the easiest way to understand these three categories is to understand their effect on gravity waves in moving water. Imagine a pebble thrown into a stream. The pebble will create a ripple in the moving water. This ripple is a gravity wave. The speed that the wave moves in a static reference frame is $c = \sqrt{gh}$, where h is the depth of water.

- If the flow is subcritical, then the gravity wave is faster than the bulk motion of the water, meaning the ripple will propagate both upstream and downstream.
- If the flow is supercritical, then the bulk motion of the water is faster than the gravity wave, meaning that the entire ripple will move downstream.
- If the flow is critical, then the gravity wave will be the exact same speed as the bulk motion of the water, meaning that the upstream edge of the ripple will remain fixed in place, while the rest of the ripple will propagate downstream.

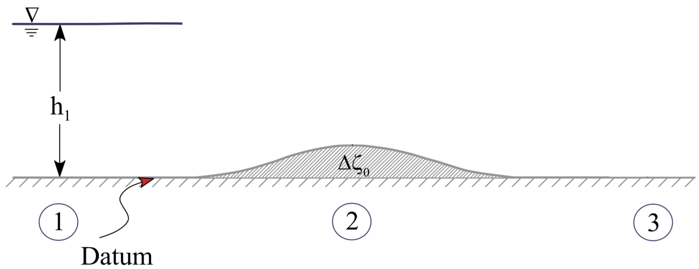


Let's return to the specific energy curve we observed earlier. For the case of critical flow ($F_r = 1$), there is one unique critical depth corresponding to a given discharge and channel geometry. However, for $F_r < 1$ or $F_r > 1$ there are always two depths on the curve corresponding to a given specific energy. The larger of these two depths corresponds to *subcritical flow* and the smaller of these two depths corresponds to *supercritical flow*.



Flow transitions

We can now return to our original discussion of flow over a smooth bottom ridge:



EXAMPLE 4.8

Problem: Consider a 3 $[m]$ wide rectangular channel. A smooth ridge is installed at the bottom channel. This ridge is 0.3 $[m]$ high at its peak. Upstream of the transition, the depth is equal to 3 $[m]$, and the velocity is measured to be 3 $[m/s]$. Assume no energy losses over the ridge or channel bottom. What is the absolute water level at the crest of the ridge?

Solution: From before, we determined that the depth must take one of the following values:

$$h_2 \in \{2.496, 1.659, -0.997\} \tag{384}$$

First, let's compute the Froude number at the upstream end of the channel.

$$F_r = \frac{u\sqrt{T}}{\sqrt{gA}} = \frac{3\sqrt{3}}{\sqrt{9.81 \cdot (3 \cdot 3)}} = 0.55 \tag{385}$$

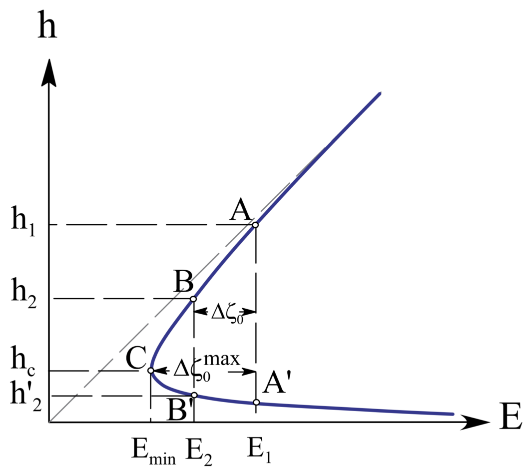
The flow upstream of the transition is thus *subcritical*. We also know that the depth does not pass through the critical depth before the transition. Thus, because the depth at the transition must be subcritical and non-negative:

$$\boxed{h_2 = 2.496\ [m]} \tag{386}$$

Thus, the absolute water depth is:

$$\boxed{h_2 + \Delta\zeta_0 = 2.796\ [m]} \tag{387}$$

How did we know that the depth is subcritical at the bump? Consider the energy diagram shown below:



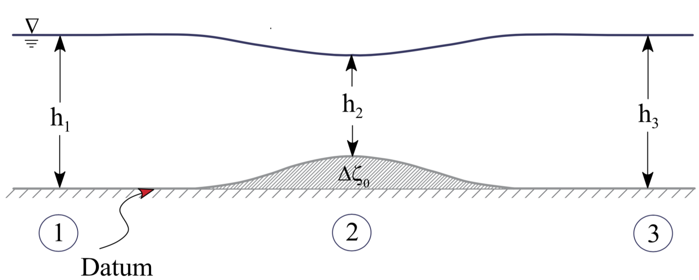
For an initially subcritical flow at depth h_1, we start at point A on the specific energy curve. The change in the bottom elevation results in a change in the specific energy that moves us along the curve. The energy balance between the upstream point (1) and the bump (2) is given by:

$$h_1 + \frac{u_1^2}{2g} = h_2 + \frac{u_2^2}{2g} + \Delta\zeta_0 \qquad (388)$$

Which can be rewritten in terms of the specific energies:

$$E_1 = E_2 + \Delta\zeta_0 \qquad (389)$$

Thus, as can be seen on the diagram, the change in specific energy from point (1) to point (2) is $\Delta\zeta_0$. Thus, we move on the curve from point A to either point B or point B', with B corresponding to subcritical flow and point B' corresponding to supercritical flow. To reach point B' however, we must pass through the critical depth. Reaching the critical depth would require us to pass through a change in bottom elevation of $\Delta\zeta_{max}$, which is the largest possible bump that would enable flow through the channel at the given discharge Q and channel geometry. Because (2) corresponds to the peak of the bump, it is not possible for a larger $\Delta\zeta_{max}$ to be reached beforehand. Thus, it is not possible for the flow at the peak of the ridge to be supercritical. The flow at h_2 must thus be subcritical, and so we select the largest root of the cubic equation.



Example 4.9

Problem: Consider the same flow conditions as in the previous problem; however, we will now design the bottom ridge such that critical flow occurs at the crest of the ridge. Find the height of the ridge $\Delta\zeta_{max}$ needed to produce these conditions.

Solution: Setting the Froude number to 1, we have:

$$F_r = 1 = \frac{Q^2 T}{g A^3} \tag{390}$$

$$Q^2 B = g B^3 h_c^3 \tag{391}$$

$$h_c = \left(\frac{Q^2}{g B^2}\right)^{1/3} = \left(\frac{27^2}{9.81 \cdot 3^2}\right)^{1/3} \tag{392}$$

$$h_c = 2.021\ [m] \tag{393}$$

Applying conservation of energy:

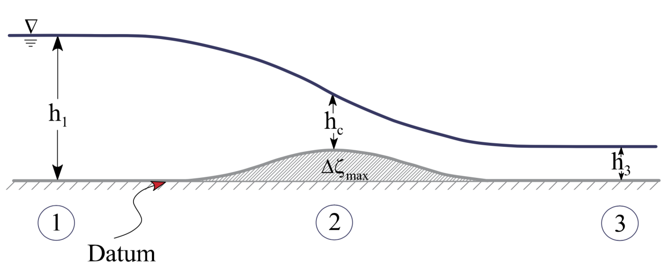
$$h_1 + \frac{u_1^2}{2g} = h_c + \frac{Q^2}{2 g B^2 h_c^2} + \Delta\zeta_0 \tag{394}$$

$$3 + \frac{3^2}{2 \cdot 9.81} = 2.021 + \frac{27^2}{2 \cdot 9.81 \cdot 3^2 \cdot 2.021^2} + \Delta\zeta_{max} \tag{395}$$

Evaluating this expression, we get:

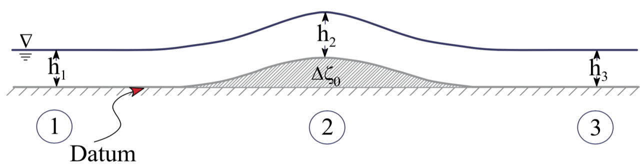
$$\boxed{\Delta\zeta_{max} = 0.427\ [m]} \tag{396}$$

The flow profile for the case where the water surface profile passes through the critical depth at the ridge is shown below.



Note that the flow is depicted as transitioning to supercritical after passing through the critical depth at the peak of the bump. Theoretically, it is possible for the flow to return to its previous subcritical state after achieving the critical depth. This condition can be achieved under carefully-controlled laboratory conditions. However, in practice, the flow will almost always become supercritical after achieving the critical depth at the transition.

For the case of a supercritical approach, the water will generally rise over the bump. If the supercritical approach flow becomes critical over the peak of the bump, it will generally return to supercritical flow on the downstream side.



Mild and steep slopes

As discussed earlier, open channel flow will tend towards uniform flow when traveling over long distances at relatively constant slopes. The depth at which uniform flow occurs is called the normal depth. We can now characterize whether uniform flow is subcritical or supercritical. A slope that maintains subcritical flow is called a *mild slope* whereas a slope that maintains supercritical flow is called a *steep slope*.

EXAMPLE 4.10

Problem: Consider the rectangular channel from before carrying uniform flow, with bottom slope $S_0 = 0.0004$, bottom width $B = 8[ft]$, normal depth of flow 2 $[ft]$, and roughness $n = 0.015$. Is the flow subcritical or supercritical under uniform flow conditions?

Solution: Before, using Manning's equation, we computed the discharge as 38.5 [cfs]:

$$Q = \frac{\phi}{n} A R^{2/3} S_0^{1/2} = 38.5 \; [cfs] \tag{397}$$

Computing the Froude number:

$$F_r = \frac{u\sqrt{T}}{\sqrt{gA}} = \frac{Q\sqrt{T}}{A\sqrt{gA}} \tag{398}$$

$$= \frac{Q\sqrt{B}}{Bh\sqrt{gBh}} = \frac{Q}{Bh\sqrt{gh}} = \frac{38.5}{8(2)\sqrt{32.2 \cdot 2}} = 0.3 \tag{399}$$

The Froude number is less than 1, and thus the flow is subcritical.

Example 4.11

Problem: Now, change the slope to $S_o = 0.01$. Is the flow subcritical or supercritical?

Solution: Recomputing the discharge under the new slope:

$$Q = \frac{\phi}{n} A R^{2/3} S_0^{1/2} \tag{400}$$

$$= \frac{1.49}{0.015}(8)(2)\left[\frac{(8)(2)}{8+2(2)}\right]^{2/3}(0.01)^{1/2} \tag{401}$$

$$= 192.5\ [cfs] \tag{402}$$

Computing the Froude number:

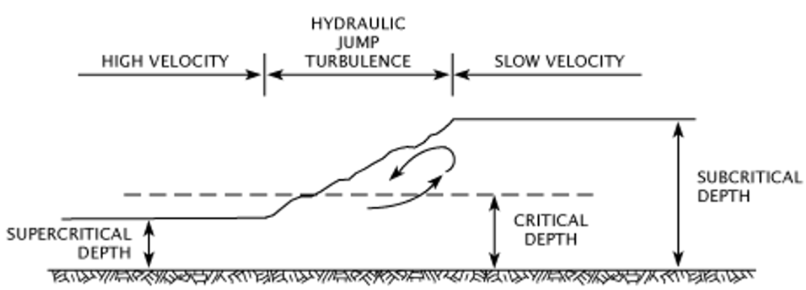
$$F_r = \frac{u\sqrt{T}}{\sqrt{gA}} = \frac{Q}{Bh\sqrt{gh}} = \frac{192.5}{8(2)\sqrt{32.2 \cdot 2}} = 1.5 \tag{403}$$

The Froude number is greater than 1, and thus the flow is supercritical.

Hydraulic jumps

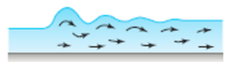
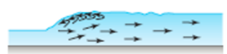
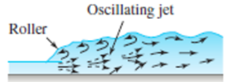
Hydraulic jumps occur when open channel flow transitions from supercritical to subcritical flow. These transitions can occur in a variety of situations, including:

- Transitions from a steep slope to a mild slope.
- Rapid flow from a sluice gate that subsequently becomes subcritical.
- Rapid flow at the foot of a spillway that subsequently becomes subcritical.



A hydraulic jump is characterized by supercritical flow at the upstream end, subcritical flow at the downstream end, and a region of highly turbulent flow in the middle. The flow passes through the critical depth in this turbulent transitional region of flow. The turbulent eddies in this region cause a large amount of energy dissipation. The exact shape of the hydraulic jump depends on the upstream Froude number, as shown below.

Table 10.2 Hydraulic Jumps in Horizontal Rectangular Channels

Upstream Fr	*Type*	*Description*	
1.0–1.7	Undular	Ruffled or undular water surface; surface rollers form near Fr = 1.7	
1.7–2.5	Weak	Prevailing smooth flow; low energy loss	
2.5–4.5	Oscillating	Intermittent jets from bottom to surface, causing persistent downstream waves	
4.5–9.0	Steady	Stable and well-balanced; energy dissipation contained in main body of jump	
>9.0	Strong	Effective, but with rough, wavy surface downstream	

Source: Adapted with permission from Chow, 1959. (Based on Chow, 1959)

Hydraulic jumps often occur in natural channels. In designed systems, they are often created artificially to reduce flow velocity and thus reduce the potential for erosion downstream.

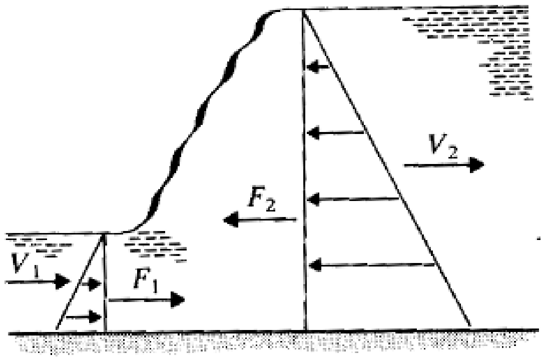
Computing the sequent depth of a hydraulic jump

Within a hydraulic jump, turbulent eddies dissipate energy, meaning that the specific energy downstream of the jump will be smaller than the specific energy upstream. Thus, it is not possible to use the specific energy diagram to determine the sequent depth after the jump. Rather, to characterize hydraulic jumps, we can instead use the conservation of momentum.

Let's consider a hydraulic jump in a horizontal channel. Recall that the momentum equation corresponds to Newton's second law applied over our control volume of water:

$$\frac{d(mu)}{dt} = \sum F \tag{404}$$

The change in momentum is equal to the time rate of change of momentum within the control volume plus the momentum flux through the control volume.



The sum of forces is likewise equal to the net hydrostatic force plus the component of the gravitational force in the direction of flow plus the frictional force which acts opposite to the direction of flow.

$$\frac{d(mu)}{dt} + u_2\dot{m}_2 - u_1\dot{m}_1 = F_h + F_g + F_f \tag{405}$$

Here, we consider the system to be at steady state and thus the time rate of momentum change is zero:

$$\cancelto{0}{\frac{d(mu)}{dt}} + u_2\dot{m}_2 - u_1\dot{m}_1 = F_h + F_g + F_f \tag{406}$$

Likewise, because the channel is horizontal, the component of the gravitational force in the direction of flow $F_g = mg\sin\theta$ is zero. The distance is also short enough that the frictional force is negligible. Thus, we have that the net momentum flux is equal to the net hydrostatic force:

$$\cancelto{0}{\frac{d(mu)}{dt}} + u_2\dot{m}_2 - u_1\dot{m}_1 = F_h + \cancelto{0}{F_g} + \cancelto{0}{F_f} \tag{407}$$

The net hydrostatic force is equal to the hydrostatic force at the upstream end minus the hydrostatic force at the downstream end:

$$F_h = \rho g\bar{h}_2 A_1 - \rho g\bar{h}_2 A_2 \tag{408}$$

Where $\bar{h}$ is the centroidal depth. Thus, we have:

$$u_2\dot{m}_2 - u_1\dot{m}_1 = \rho g\bar{h}_1 A_1 - \rho g\bar{h}_2 A_2 \tag{409}$$

$$\rho Q(u_2 - u_1) = \rho g(\bar{h}_1 A_1 - \bar{h}_2 A_2) \tag{410}$$

And thus, we can posit the following relationship between the hydraulic geometry and velocity:

General equation for depths before and after hydraulic jump

$$\boxed{Q^2\left(\frac{1}{A_2} - \frac{1}{A_1}\right) = g(\bar{h}_1 A_1 - \bar{h}_2 A_2)} \tag{411}$$

Hydraulic jumps in rectangular channels

Now, for the sake of simplicity, we will assume a rectangular channel. For a rectangular channel, the centroidal depth $\bar{h} = \frac{1}{2}h$, and $A = Bh$. Thus, we have:

$$Q^2\left(\frac{1}{Bh_2} - \frac{1}{Bh_1}\right) = \frac{gB}{2}(h_1^2 - h_2^2) \quad (412)$$

$$Q^2\left(\frac{h_1 - h_2}{Bh_1h_2}\right) = \frac{gB}{2}(h_1^2 - h_2^2) \quad (413)$$

$$\frac{Q^2}{gB^2h_1}\left(\frac{h_1 - h_2}{h_2}\right) = \frac{1}{2}(h_1 - h_2)(h_1 + h_2) \quad (414)$$

$$\frac{Q^2}{gB^2h_1} = \frac{h_2}{2}(h_1 + h_2) \quad (415)$$

$$\frac{Q^2}{gB^2h_1}\frac{Bh_1^2}{Bh_1^2} = \frac{h_2}{2}(h_1 + h_2) \quad (416)$$

$$\frac{Q^2B}{gA^3}h_1^2 = \frac{h_2}{2}(h_1 + h_2) \quad (417)$$

$$F_{r1}^2h_1^2 = \frac{h_2}{2}(h_1 + h_2) \quad (418)$$

$$F_{r1}^2 = \frac{h_2}{2h_1^2}(h_1 + h_2) \quad (419)$$

$$F_{r1}^2 = \frac{1}{2}\left(\frac{h_2}{h_1} + \frac{h_2^2}{h_1^2}\right) \quad (420)$$

Rearranging, note that we have a polynomial in h_2/h_1:

$$\frac{1}{2}\left(\frac{h_2^2}{h_1^2}\right) + \frac{1}{2}\left(\frac{h_2}{h_1}\right) - F_{r1}^2 = 0 \quad (421)$$

$$\left(\frac{h_2^2}{h_1^2}\right) + \left(\frac{h_2}{h_1}\right) - 2F_{r1}^2 = 0 \quad (422)$$

We can find the roots of this polynomial using the quadratic formula:

$$\frac{h_2}{h_1} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (423)$$

$$\frac{h_2}{h_1} = \frac{-1 \pm \sqrt{1^2 + 4(2F_{r1}^2)}}{2} \quad (424)$$

$$\frac{h_2}{h_1} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 8F_{r1}^2} \quad (425)$$

Discarding the negative root, we have:

Equation for depths before and after hydraulic jump in a rectangular channel

$$\boxed{\frac{h_2}{h_1} = \frac{1}{2}\left(\sqrt{1 + 8F_{r1}^2} - 1\right)} \quad (426)$$

EXAMPLE 4.12

Problem: A 10-ft wide rectangular channel carries 500 cfs of water at a 2-ft depth before entering a jump. Compute the downstream water depth and the critical depth.

Solution: Using the sequent depth equation for a rectangular channel:

$$\frac{h_2}{h_1} = \frac{1}{2}\left(\sqrt{1+8F_{r1}^2} - 1\right) \tag{427}$$

$$h_2 = \frac{h_1}{2}\left(\sqrt{1+8\frac{Q^2B}{gA_1^3}} - 1\right) \tag{428}$$

$$h_2 = \frac{h_1}{2}\left(\sqrt{1+8\frac{Q^2}{gB^2h_1^3}} - 1\right) \tag{429}$$

Plugging in numbers we have:

$$h_2 = \frac{2}{2}\left(\sqrt{1+8\frac{500^2}{32.2(10^2)(2^3)}} - 1\right) \tag{430}$$

Thus:

$$\boxed{h_2 = 7.88\ [ft]} \tag{431}$$

The critical depth corresponds to the conditions:

$$1 = \frac{Q^2B}{gA^3} = \frac{Q^2B}{gB^3h^3} \tag{432}$$

$$h^3 = \frac{Q^2}{gB^2} \tag{433}$$

$$h = \left(\frac{Q^2}{gB^2}\right)^{1/3} \tag{434}$$

$$h = \left(\frac{500^2}{32.2(10^2)}\right)^{1/3} \tag{435}$$

Thus:

$$\boxed{h_c = 4.27\ [ft]} \tag{436}$$

EXAMPLE 4.13

Problem: For the previous question, what is the energy loss through the jump?

Solution: We can determine the energy loss by applying an energy balance between the upstream and downstream ends:

$$h_1 + \frac{u_1^2}{2g} = h_2 + \frac{u_2^2}{2g} + h_L \tag{437}$$

$$h_L = h_1 + \frac{u_1^2}{2g} - h_2 + \frac{u_2^2}{2g} \tag{438}$$

Because the channel is rectangular, we have:

$$u_1 = \frac{Q}{A_1} = \frac{Q}{Bh_1} = \frac{500\ [cfs]}{(10\ [ft])(2\ [ft])} = 25\ [ft/s] \tag{439}$$

$$u_2 = \frac{Q}{A_2} = \frac{Q}{Bh_2} = \frac{500\ [cfs]}{(10\ [ft])(7.88\ [ft])} = 6.345\ [ft/s] \tag{440}$$

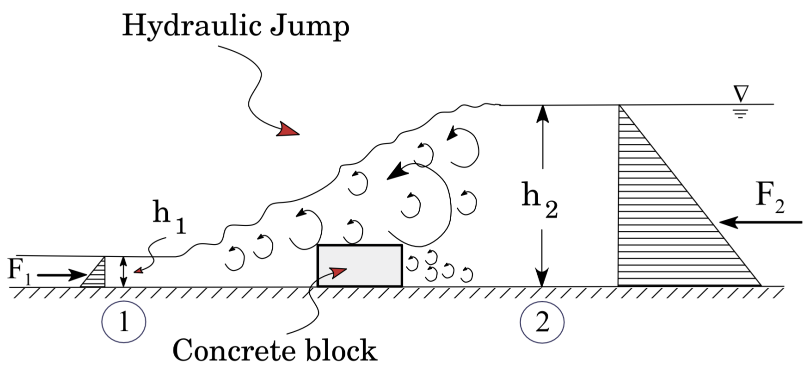
Thus we have:

$$h_L = 2 + \frac{25^2}{2(32.2)} - 7.88 - \frac{6.345^2}{2(32.2)} \tag{441}$$

$$\boxed{h_L = 3.2\ [ft]} \tag{442}$$

Energy dissipators

Hydraulic jumps are often facilitated through the use of concrete blocks known as energy dissipators.



These blocks induce a reaction force on the water given by Newton's drag law:

$$F_{block} = \frac{1}{2}\rho C_D A u_1^2 \tag{443}$$

The momentum balance thus becomes:

$$\rho Q(u_2 - u_1) = \rho g(\bar{h}_1 A_1 - \bar{h}_2 A_2) - \frac{1}{2}\rho C_D A_1 u_1^2 \tag{444}$$

This reaction force and the energy dissipation it induces helps to reduce the sequent depth of the hydraulic jump.

EXAMPLE 4.14

Problem: In the design of stilling basins, concrete blocks are commonly placed on the bottom to assist the formation of the hydraulic jump, and to ensure that it is totally contained within the basin for a wide range of flow rates. The blocks are usually staggered on two rows, thus their frontal area covers the entire width of the channel while allowing for free flow around them. The force exerted on the blocks can thus be computed assuming that the frontal area is equal to the block height times the channel width. Such an arrangement leads to a drag coefficient CD = 0.3. If the discharge is 100 m^3/s, and the depth upstream of the blocks is 0.6 m, find the downstream depth required to trap the jump in the basin by first assuming there are no blocks, and then assuming that there are two rows of blocks 0.6 m high. Assume that the width of the channel is equal to 10 m.

Solution: Assuming there are no blocks, we can compute the sequent depth using the sequent depth equation:

$$\frac{h_2}{h_1} = \frac{1}{2}\left(\sqrt{1+8F_{r1}^2} - 1\right) \tag{445}$$

$$h_2 = \frac{h_1}{2}\left(\sqrt{1+8\frac{Q^2}{gB^2h_1^3}} - 1\right) \tag{446}$$

Plugging in numbers we have:

$$h_2 = \frac{0.6}{2}\left(\sqrt{1+8\frac{100^2}{9.81(10^2)(0.6^3)}} - 1\right) \tag{447}$$

Thus, with no blocks we have:

$$\boxed{h_2 = 5.54\ [m]} \tag{448}$$

To compute the sequent depth with blocks, we return to the momentum equation:

$$\rho Q(u_2 - u_1) = \rho g(\bar{h}_1 A_1 - \bar{h}_2 A_2) - \frac{1}{2}\rho C_D A_1 u_1^2 \tag{449}$$

Now, because the channel is rectangular, we can simplify the momentum equation:

$$\rho Q^2\left(\frac{1}{Bh_2} - \frac{1}{Bh_1}\right) = \frac{\rho g B}{2}(h_1^2 - h_2^2) - \frac{1}{2}\rho C_D B h_1 \frac{Q^2}{B^2h_1^2} \tag{450}$$

Substituting known values, we have:

$$(100)^2\left(\frac{1}{10h_2}-\frac{1}{10(0.6)}\right)=\frac{9.81(10)}{2}(0.6^2-h_2^2)-\frac{1}{2}(0.3)(10)(0.6)\frac{(100)^2}{10^2(0.6^2)} \tag{451}$$

$$\frac{1000}{h_2}-1434.3+49.05h_2^2=0 \tag{452}$$

$$49.05h_2^3-1434.3h_2+1000=0 \tag{453}$$

Solving the roots of this polynomial and selecting the largest value (corresponding to the subcritical depth, we get:

$$\boxed{h_2=5.02\ [m]} \tag{454}$$

Thus, the blocks reduce the sequent depth by about 0.5 [m].

Specific force

Recall that we can characterize the behavior of the hydraulic jump using a momentum balance:

$$\rho Q(u_2-u_1)=\rho g(\bar{h}_1A_1-\bar{h}_2A_2) \tag{455}$$

$$\rho Qu_1+\rho g\bar{h}_1A_1=\rho Qu_2+\rho g\bar{h}_2A_2 \tag{456}$$

We can define the specific force as:

Specific force

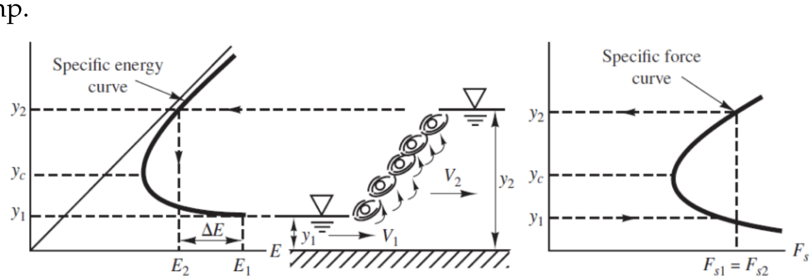
$$\boxed{F_s=\rho Qu+\rho g\bar{h}A} \tag{457}$$

For a rectangular channel, this expression for the specific force can be simplified, given that $\bar{h}=h/2$ and $A=Bh$:

Specific force for a rectangular channel

$$F_s=\rho Qu+\rho g\frac{h^2}{2} \tag{458}$$

Just like the specific energy, we can create the plot of the specific force across the transition. Unlike the specific energy, the specific force remains the same on the downstream end of the hydraulic jump.



Hydraulic control structures

Orifices

An *orifice* consists of a submerged opening in a tank or otherwise quiescent body of water through which water is driven by pressure. Orifices can be found in many contexts within water resources engineering, including:

- Outlet structures for stormwater detention basins
- Curb inlets
- Storm sewer pipe outfalls into natural channels
- Gates and outlet structures in dams

The velocity of the fluid jet can be derived by applying an energy balance between the surface of the tank and a point in the jet of water slightly downstream of the opening. Applying the energy equation:

$$\frac{p_1}{\gamma} + z_1 + \frac{u_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{u_2^2}{2g} + h_L \tag{459}$$

The tank surface and the water jet are both at atmospheric pressure. Moreover, the velocity at the tank surface is zero. Finally, we will assume that head losses are negligible. Thus, the equation can be rewritten:

$$\cancelto{0}{\frac{p_1}{\gamma}} + z_1 + \cancelto{0}{\frac{u_1^2}{2g}} = \cancelto{0}{\frac{p_2}{\gamma}} + z_2 + \frac{u_2^2}{2g} + \cancelto{0}{h_L} \tag{460}$$

Let $h = z_1 - z_2$ be the height of the tank surface above the orifice. Thus we have:

$$h = \frac{u_2^2}{2g} \tag{461}$$

Thus, the velocity of flow for the jet exiting the orifice depends only on the depth of water above the orifice:

Velocity of flow for water jet exiting an orifice

$$\boxed{u = \sqrt{2gh}} \tag{462}$$

The flow rate is determined by multiplying the velocity by the cross-sectional area of the jet of water. Let A_o be the cross sectional area of the opening. Then we can define $A = CA_o$ as the cross sectional area of the fluid jet at its narrowest point (called the *vena contracta*). Here, C is the orifice coefficient that defines the ratio of the cross-sectional area of the vena contracta to the cross-sectional area of the orifice opening. The flow rate through an orifice can thus be defined as:

Flow rate through an orifice

$$\boxed{Q = CA_o\sqrt{2gh}} \tag{463}$$

Where the value of C depends on the shape of the orifice opening. For circular orifices, a value of $C \approx 0.6$ is typical.

Weirs

A *weir* is a hydraulic control structure that is designed to alter flow by forcing a transition from subcritical to supercritical flow.

Weir terminology

The following terminology will be used when discussing weirs:

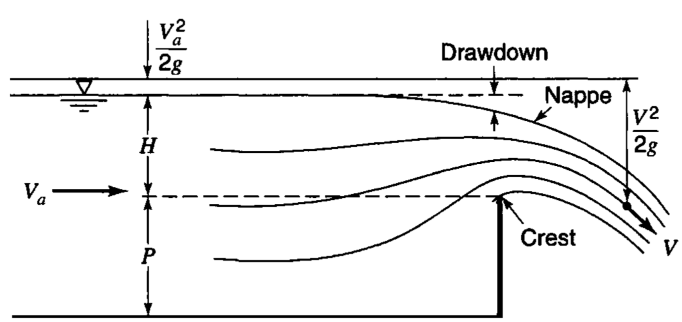
Notch: the opening which water flows through

Crest: the edge over which water flows

Nappe: the overflowing sheet of water

Length: the width of the weir crest

Height: the height of the weir crest



Types of weirs

Weirs are divided into two major types:

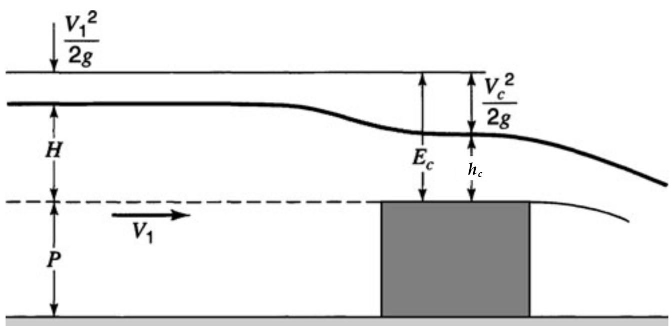
Broad-crested weir: consists of a broad obstruction at the bottom of a channel. Broad-crested weirs create free overflows such that the critical depth occurs on the crest of the weir.

Sharp-crested weir: consists of a sharp plate mounted within the channel. Water flows through a notch in the plate. Critical flow is generally achieved just before the crest.

Ideal broad-crested weirs

First, let's consider an idealized broad-crested weir. We can derive a relationship between the flow rate and the depth over the weir by assuming:

- Critical flow is achieved over the crest of the weir
- No energy losses due to friction or contraction/expansion losses
- A rectangular channel
- The kinetic energy head upstream of the weir is negligible



Applying the energy equation between the upstream section and the critical section, we have:

$$h_1 + \frac{u_1^2}{2g} + z_1 = h_2 + \frac{u_2^2}{2g} + z_2 + h_L \tag{464}$$

Neglecting the upstream velocity head and the head loss:

$$h_1 + \cancelto{0}{\frac{u_1^2}{2g}} + z_1 = h_2 + \frac{u_2^2}{2g} + z_2 + \cancelto{0}{h_L} \tag{465}$$

$$h_1 = h_c + \frac{u_2^2}{2g} + \Delta z \tag{466}$$

$$h_1 = h_c + \frac{Q^2}{2gB^2h_c^2} + \Delta z \tag{467}$$

Under critical flow, we have that the Froude number is equal to 1, and thus the square of the Froude number is also equal to one. For a rectangular channel we have:

$$F_r^2 = 1 = \frac{u^2 T}{gA} = \frac{Q^2 B}{gB^3 h_c^3} \tag{468}$$

$$h_c = \left(\frac{Q^2}{gB^2}\right)^{1/3} \tag{469}$$

Substituting back into the energy equation, we have:

$$h_1 = \left(\frac{Q^2}{gB^2}\right)^{1/3} + \frac{Q^2}{2gB^2}\left(\frac{Q^2}{gB^2}\right)^{-2/3} + \Delta z \tag{470}$$

$$h_1 = \left(\frac{Q^2}{gB^2}\right)^{1/3} + \frac{1}{2}\left(\frac{Q^2}{gB^2}\right)^{1/3} + \Delta z \tag{471}$$

$$h_1 = \frac{3}{2}\left(\frac{Q^2}{gB^2}\right)^{1/3} + \Delta z \tag{472}$$

Rearranging for Q:

$$\left(\frac{Q^2}{gB^2}\right)^{1/3} = \frac{2}{3}(h_1 - \Delta z) \tag{473}$$

$$Q^2 = gB^2\left[\frac{2}{3}(h_1 - \Delta z)\right]^3 \tag{474}$$

$$\tag{475}$$

Thus:

Discharge equation for an ideal broad-crested rectangular weir

$$\boxed{Q = B\sqrt{g\left[\frac{2}{3}(h_1 - \Delta z)\right]^3}} \tag{476}$$

Often for succinctness, this equation will be written:

Abbreviated form of the weir equation for an ideal broad-crested rectangular weir

$$\boxed{Q = C_w B H_w^{3/2}} \tag{477}$$

Where H_w is the head above the weir crest ($H = h_1 - \Delta z$) and C_w is a weir coefficient that groups together the constants. The value of this weir coefficient depends on the unit system:

- $C_w = 1.70$ for SI units
- $C_w = 3.09$ for US units

Example 4.15

Problem: A 3.05 [m] wide frictionless broad-crested weir rises 1.10 [m] above the channel bottom. The water depth upstream of the weir is 1.89 [m]. Determine the discharge in the channel and the velocity of the water going over the weir. Assume then upstream velocity head is negligible.

Solution: Applying the ideal rectangular weir equation:

$$Q = C_w B H_w^{3/2} = 1.70(3.05)(1.89 - 1.10)^{3/2} \tag{478}$$

$$\boxed{Q = 3.66\ [m^3/s]} \tag{479}$$

To get the velocity, we need to know the critical depth. For a rectangular channel we have:

$$h_c = \left(\frac{Q^2}{gB^2}\right)^{1/3} = \left(\frac{3.66^2}{9.81(3.05)^2}\right)^{1/3} = 0.528\ [m] \tag{480}$$

Thus, the velocity over the weir is given by:

$$u_c = \frac{Q}{Bh_c} = \frac{3.66}{3.05(0.528)} \tag{481}$$

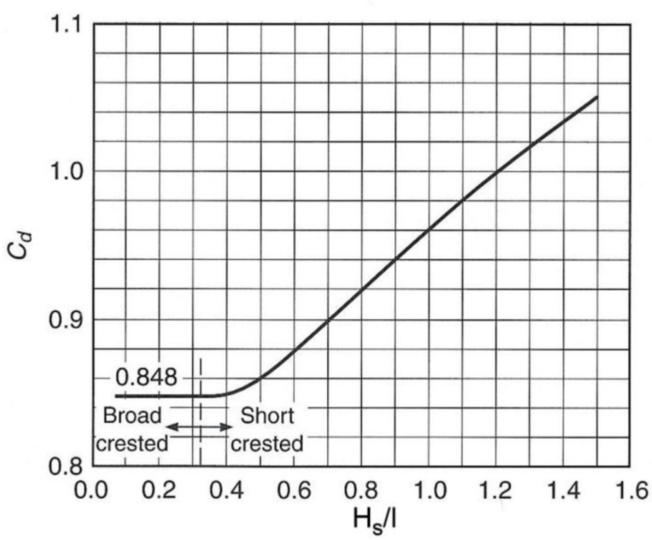
$$\boxed{u_c = 2.28\ [m/s]} \tag{482}$$

Non-ideal weirs

In reality, the behavior of weirs will deviate from the ideal case due to energy losses resulting from converging streamlines and the contribution of the upstream velocity head. To account for these discrepancies, we introduce discharge coefficient C_d and velocity coefficient C_v:

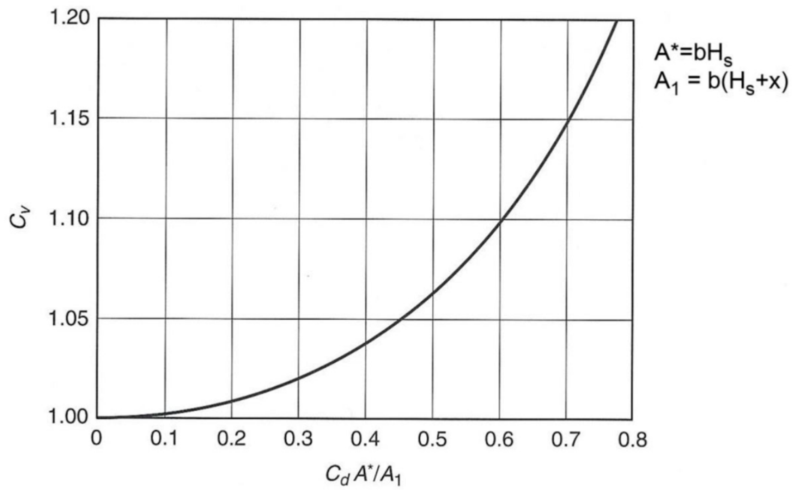
$$\boxed{Q = C_d C_v B \sqrt{g \left[\frac{2}{3}(h_1 - \Delta z)\right]^3}} \tag{483}$$

The coefficient of discharge is given by the following chart:



(b) Coefficient of Discharge for $H_s/(H_s+x) \leq 0.35$

The velocity coefficient is likewise given by the following chart:



EXAMPLE 4.16

Problem: A weir 0.95 m in height and 0.6 m in length is built across the base of a 3.8 m wide rectangular channel. The water depth just upstream of the weir is 1.4 m. Compute the discharge (cms), velocity (m/s), and depth (m) over the weir, when (i) neglecting friction losses and kinetic energy upstream, and (ii) Considering non-ideal flow conditions

Solution: First, considering the case of ideal flow conditions, we have:

$$Q = C_w B H_w^{2/3} = 1.7(0.6)(1.4 - 0.95)^{3/2} \tag{484}$$

$$\boxed{Q = 1.95\ [m^3/s]} \tag{485}$$

The depth over the weir is given by the critical depth:

$$h_c = \left(\frac{Q^2}{gB^2}\right)^{1/3} = \left(\frac{1.95^2}{9.81(3.8)^2}\right)^{1/3} \tag{486}$$

$$\boxed{h_c = 0.3\ [m]} \tag{487}$$

The velocity is then given by dividing the discharge by the flow area:

$$u_c = \frac{Q}{Bh_c} = \frac{1.95}{3.8(0.3)} \tag{488}$$

$$\boxed{u_c = 1.71\ [m/s]} \tag{489}$$

Now, considering a non-ideal weir, we have:

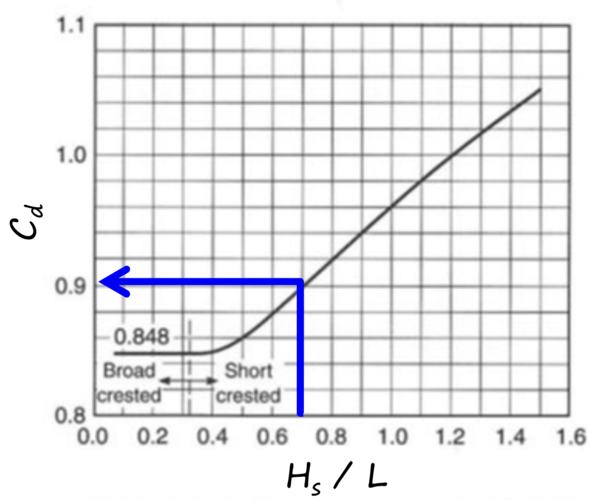
$$Q = C_d C_v B \sqrt{g \left[\frac{2}{3}(h_1 - \Delta z) \right]^3} \tag{490}$$

To determine the discharge coefficient C_d, we first determine the ratio of the weir head to the weir head plus the height of the weir:

$$\frac{H_w}{H_w + \Delta z} = \frac{0.45}{0.45 + 0.95} = 0.32 < 0.35 \tag{491}$$

The ratio is less than 0.35 and thus we can determine the discharge coefficient empirically using our chart. Taking the ratio of the weir head to the width of the weir, we have:

$$\frac{H_w}{B} = \frac{0.45}{0.6} = 0.75 \tag{492}$$



(b) Coefficient of Discharge for Hs/(Hs+x)≤ 0.35

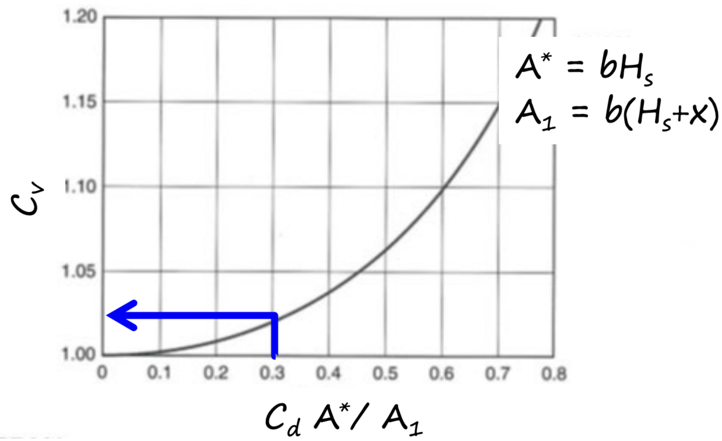
Using our chart, this value corresponds to a discharge coefficient of $C_d = 0.9$. Now, we can determine the approach velocity correction coefficient.

$$A^* = BH_w = 1.71\ [m^2] \tag{493}$$

$$A_1 = B(H_w + \Delta z) = 5.32\ [m^2] \tag{494}$$

$$C_d A^* / A_1 = 0.29 \tag{495}$$

Using our chart, this value corresponds to a velocity coefficient of $C_v = 1.02$.



Thus, we have:

$$Q = C_d C_v B \sqrt{g \left[\frac{2}{3}(h_1 - \Delta z) \right]^3} = (0.9)(1.02)(1.95) \tag{496}$$

$$\boxed{Q = 1.8 \; [m^3/s]} \tag{497}$$

$$h_c = \left(\frac{Q^2}{gB^2} \right)^{1/3} = \left(\frac{1.8^2}{9.81(3.8)^2} \right)^{1/3} \tag{498}$$

$$\boxed{h_c = 0.284 \; [m]} \tag{499}$$

$$u_c = \frac{Q}{Bh_c} = \frac{1.8}{3.8(0.3)} \tag{500}$$

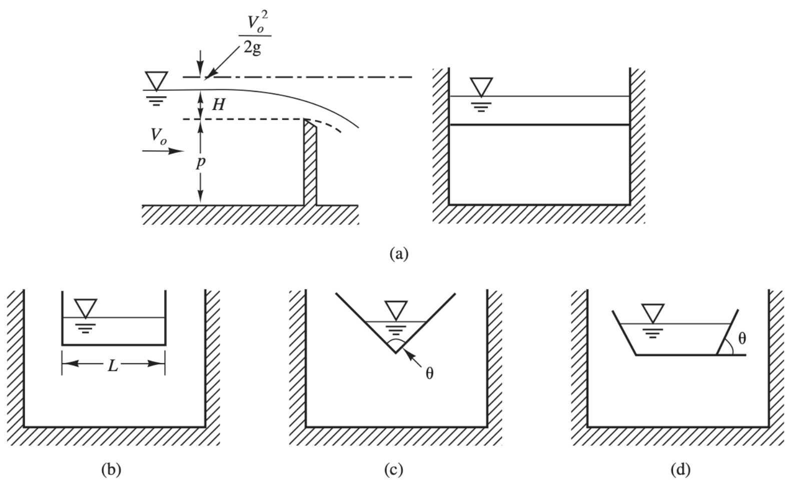
$$\boxed{u_c = 1.66 \; [m/s]} \tag{501}$$

Sharp-crested weirs

A sharp-crested weir consists of a plate mounted normal to the direction of flow. Sharp-crested weirs come in four basic types:

- Uncontracted (rectangular) horizontal
- Contracted horizontal
- V-notch weirs
- Trapezoidal

The crest can extend across the entire width of the channel or not. Notched weirs are used in wider channels with slower flow rates. V-notch weirs are used where very low flow rates may occur.



The discharge equations for an ideal sharp-crested rectangular weir is given as follows:

Discharge equation for ideal sharp-crested rectangular uncontracted weir

$$Q = B\frac{2}{3}\sqrt{2g}H_w^{3/2} \tag{502}$$

Where Q is the weir discharge, B is the weir length and H_w is the head over the weir.

The discharge for a V-notch weir is given as follows:

Discharge equation for ideal sharp-crested V-notch weir

$$Q = \frac{8}{15}\tan\frac{\theta}{2}\sqrt{2g}H_w^{3/2} \tag{503}$$

Where Q is the weir discharge, θ is the notch angle and H_w is the head over the weir.

Gradually-varied flow

Gradually-varied flow occurs when the depth and velocity vary gradually along the length of an open channel, typically with the slope of the water surface being much smaller than unity $|\partial h/\partial x| < 1$. The general equation for gradually-varied flow can be derived from the momentum equation and the continuity equation:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{504}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial (Qu)}{\partial x} + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f\right) = 0 \tag{505}$$

For a system at steady-state, we have:

$$\cancelto{0}{\frac{\partial A}{\partial t}} + \frac{\partial Q}{\partial x} = 0 \tag{506}$$

$$\cancelto{0}{\frac{\partial Q}{\partial t}} + \frac{\partial (Qu)}{\partial x} + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f\right) = 0 \tag{507}$$

By the product rule, the inertial term can be expanded:

$$\frac{\partial Q}{\partial x} = 0 \tag{508}$$

$$u\frac{\partial Q}{\partial x} + Q\frac{\partial u}{\partial x} + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f\right) = 0 \tag{509}$$

Substituting the first equation into the second:

$$Q\frac{\partial u}{\partial x} + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f\right) = 0 \tag{510}$$

Dividing by gA:

$$\frac{u}{g}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} - S_0 + S_f = 0 \tag{511}$$

The inertial term can be expressed as:

$$\frac{\partial u}{\partial x} = \frac{\partial (Q/A)}{\partial x} \tag{512}$$

By the quotient rule:

$$\frac{\partial(Q/A)}{\partial x} = \frac{\frac{\partial Q}{\partial x}A - Q\frac{\partial A}{\partial x}}{A^2} = -\frac{Q}{A^2}\frac{\partial A}{\partial x} \tag{513}$$

By the chain rule:

$$\frac{\partial A}{\partial x} = \frac{\partial A}{\partial h}\frac{\partial h}{\partial x} = T\frac{\partial h}{\partial x} \tag{514}$$

Thus, the inertial term reduces to:

$$\frac{\partial u}{\partial x} = -\frac{QT}{A^2}\frac{\partial h}{\partial x} \tag{515}$$

Returning to the momentum equation and substituting our new expression for the inertial term:

$$-\frac{u}{g}\frac{QT}{A^2}\frac{\partial h}{\partial x} + \frac{\partial h}{\partial x} - S_0 + S_f = 0 \tag{516}$$

$$\left(-\frac{u}{g}\frac{QT}{A^2} + 1\right)\frac{\partial h}{\partial x} - S_0 + S_f = 0 \tag{517}$$

$$\left(1 - \frac{u^2 T}{gA}\right)\frac{\partial h}{\partial x} - S_0 + S_f = 0 \tag{518}$$

Thus, the water surface is given by:

$$\frac{\partial h}{\partial x} = \frac{S_0 - S_f}{1 - \frac{u^2 T}{gA}} \tag{519}$$

Noting that $F_r^2 = \frac{u^2 T}{gA}$, the gradually-varied flow equation becomes:

Gradually-varied flow equation

$$\boxed{\frac{\partial h}{\partial x} = \frac{S_0 - S_f}{1 - F_r^2}} \tag{520}$$

Where $\frac{\partial h}{\partial x}$ is the slope of the water surface, S_0 is the bottom slope, S_f is the friction slope, and F_r is the Froude number. Note that the friction slope can be approximated using Manning's equation by assuming that the frictional losses are similar to what they would be under uniform flow. From Manning's equation under uniform flow conditions, we have that:

$$Q = \frac{\phi}{n}AR^{2/3}\sqrt{S_f} \tag{521}$$

$$\sqrt{S_f} = \frac{nQ}{\phi R^{2/3} A} \tag{522}$$

Thus, for gradually-varying flow conditions:

Approximate friction slope for gradually-varying flow conditions

$$\boxed{S_f \approx \frac{n^2 Q^2}{\phi^2 R^{4/3} A^2}} \tag{523}$$

Gradually-varied flow profiles

Water surface profiles can be constructed by numerically integrating the gradually-varied flow equation.

$$\frac{\partial h}{\partial x} = \frac{S_0 - S_f}{1 - F_r^2} \tag{524}$$

Let's approximate the derivative dh/dx with a first-order difference. Then we have:

$$\frac{h_{k+1} - h_k}{\Delta x} = \frac{S_0 - S_f}{1 - F_r^2} \tag{525}$$

From before, we have:

$$S_f = \frac{n^2 Q^2}{\phi^2 R^{4/3}(h) A^2(h)} \tag{526}$$

$$F_r^2 = \frac{Q^2 T(h)}{g A^3(h)} \tag{527}$$

Where A, T and R are all functions of the depth h. Thus, we have:

$$\frac{h_{k+1} - h_k}{\Delta x} = \frac{S_0 - \frac{n^2 Q^2}{\phi^2 R^{4/3}(h) A^2(h)}}{1 - \frac{Q^2 T(h)}{g A^3(h)}} \tag{528}$$

We can use the following *explicit scheme* to compute the depth at the next element in sequence h_{k+1} using the depth at the current element h_k:

Explicit scheme for computing gradually-varied flow profile

$$\boxed{h_{k+1} = h_k + \Delta x \left(\frac{S_0 - S_f(h_k)}{1 - F_r^2(h_k)} \right)} \tag{529}$$

For example, for a rectangular channel with width B, we have:

$$h_{k+1} = h_k + \Delta x \left(\frac{S_0 - \frac{n^2 Q^2 (B + 2h_k)^{4/3}}{\phi^2 (B h_k)^{10/3}}}{1 - \frac{Q^2}{g B^2 h_k^3}} \right) \tag{530}$$

Note that for supercritical flow, we start upstream and proceed with the computation in the downstream direction; for subcritical flow we start downstream and proceed in the upstream direction.

EXAMPLE 4.17

Problem: A grouted-riprap, trapezoidal channel ($n = 0.0252$) with a bottom width of 4 [m] and side slopes of m = 1 carries a discharge 12.5 [cms] on a 0.001 slope. Compute the backwater curve (upstream water surface profile) created by a low dam that backs water up to a depth of 2 [m] immediately behind the dam. Report the water depths at distances of 188 [m] and 425 [m] upstream of the dam.

Solution: Using an explicit solution scheme:

```
import numpy as np
import scipy.optimize
import matplotlib.pyplot as plt

n = 0.025
m = 1.
Q = 12.5
S_0 = 0.001
B = 4
h_0 = 2
dx = 10.
g = 9.81
phi = 1.

def A(h):
   return (B + m * h) * h

def P(h):
   return B + 2 * h * np.sqrt(1 + m**2)

def T(h):
   return B + 2 * m * h

n_steps = 200
x_next = 0.
z_next = 0.
h_next = h_0
hs = [h_next]
xs = [x_next]
zs = [z_next]

for i in range(n_steps):
   h_prev = h_next
   A_k = A(h_prev)
   T_k = T(h_prev)
   P_k = P(h_prev)
   R_k = A_k / P_k
   S_f = n**2 * Q**2 / R_k**(4/3) / A_k**2 / phi**2
   Fr2 = Q**2 * T_k / g / A_k**3
   h_next = h_prev - (S_0 - S_f) * dx / (1 - Fr2)
   z_next = z_next + S_0 * dx
   x_next = x_next - dx
   hs.append(h_next)
   xs.append(x_next)
   zs.append(z_next)
```

Using this code, the depths at 188 [m] and 425 [m] are found to be 1.90 [m] and 1.82 [m], respectively.

In certain cases an *implicit scheme* is more appropriate (such as when the boundary condition is at the critical depth, and the explicit scheme would yield an undefined solution). An implicit scheme solves iteratively for h_{k+1} at each spatial step:

Implicit scheme for computing gradually-varied flow profile

$$\boxed{h_{k+1} = h_k + \Delta x \left(\frac{S_0 - S_f(h_{k+1})}{1 - F_r^2(h_{k+1})} \right)} \quad (531)$$

EXAMPLE 4.18

Problem: A rough-concrete trapezoidal channel (n = 0.0222) with a 3.5-ft bottom width, side slope m = 2, and bed slope of 0.012 discharges 185 cfs of fresh water from a reservoir. Determine the water surface profile in the discharge channel to within 2 percent of normal depth.

Solution: Using an implicit solution scheme:

```
n = 0.022
m = 2.
Q = 185
S_0 = 0.012
B = 3.5
h_0 = 2.76
dx = 2.
g = 32.2
phi = 1.49

def solve_h_next(h):
   A_k = A(h)
   T_k = T(h)
   P_k = P(h)
   R_k = A_k / P_k
   S_f = n**2 * Q**2 / R_k**(4/3) / A_k**2 / phi**2
   Fr2 = Q**2 * T_k / g / A_k**3
   h_next = h_prev + (S_0 - S_f) * dx / (1 - Fr2)
   return h - h_next

def solve_h_c(h):
   T_c = T(h)
   A_c = A(h)
   Fr2 = Q**2 * T_c / g / A_c**3
   return 1 - Fr2

result = scipy.optimize.root_scalar(solve_h_c, method='bisect',
    bracket=[1e-5, 10])
h_c = result.root
```

```
x_next = 0.
z_next = 0.
h_next = h_0
n_steps = 200
hs = [h_next]
xs = [x_next]
zs = [z_next]

for i in range(n_steps):
   h_prev = h_next
   result = scipy.optimize.root_scalar(solve_h_next,
       method='bisect', bracket=[1e-5, h_c - 1e-5])
   h_next = result.root
   z_next = z_next - S_0 * dx
   x_next = x_next + dx
   hs.append(h_next)
   xs.append(x_next)
   zs.append(z_next)
```

Gradually-varied flow with lateral inflow

The gradually-varied flow equation can be adapted to account for lateral overflow into the channel. Starting with the momentum equation and the continuity equation, we can add an exogenous lateral overflow term to the continuity equation to account for water entering the channel laterally (e.g. due to runoff or rainfall):

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q_L \tag{532}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial (Qu)}{\partial x} + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f\right) = 0 \tag{533}$$

Here, q_L refers to the lateral overflow per unit length of channel. Thus, the continuity equation now states that the rate of change of volume within a control volume of channel is equal to the lateral inflow per unit length (q_L) minus the net mass flux in the longitudinal direction ($\partial Q/\partial x$). As before, for a system at steady-state, we have:

$$\cancelto{0}{\frac{\partial A}{\partial t}} + \frac{\partial Q}{\partial x} = q_L \tag{534}$$

$$\cancelto{0}{\frac{\partial Q}{\partial t}} + \frac{\partial (Qu)}{\partial x} + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f\right) = 0 \tag{535}$$

By the product rule, the inertial term can be expanded:

$$\frac{\partial Q}{\partial x} = q_L \tag{536}$$

$$u\frac{\partial Q}{\partial x} + Q\frac{\partial u}{\partial x} + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f\right) = 0 \tag{537}$$

Substituting the first equation into the second:

$$uq_L + Q\frac{\partial u}{\partial x} + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f\right) = 0 \tag{538}$$

Dividing by gA:

$$\frac{uq_L}{gA} + \frac{u}{g}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} - S_0 + S_f = 0 \tag{539}$$

Using the same argument applied in the original derivation of the gradually-varied flow equation, this equation can be re-expressed as follows:

$$\left(1 - F_r^2\right)\frac{\partial h}{\partial x} - S_0 + S_f + \frac{uq_L}{gA} = 0 \tag{540}$$

Thus, the water surface is given by:

Gradually-varied flow equation with lateral inflow

$$\boxed{\frac{\partial h}{\partial x} = \frac{S_0 - S_f - \frac{uq_L}{gA}}{1 - F_r^2}} \tag{541}$$

Where $\frac{\partial h}{\partial x}$ is the slope of the water surface, S_0 is the bottom slope, S_f is the friction slope, q_L is the lateral overflow per unit length, u is the velocity of flow, g is the acceleration due to gravity, A is the cross-sectional area, and F_r is the Froude number.

Part VI

References

Bibliography

[1] Robert J. Houghtalen, A. Osman Akan, and Ned H. C. Hwang. *Fundamentals of Hydraulic Engineering Systems*. Fourth. Prentice Hall, 2010. ISBN: 978-0-13-601638-0.

[2] Larry W. Mays. *Water Resources Engineering*. Second. John Wiley and Sons, 2010. ISBN: 978-0-470-46064-1.